LIE ALGEBRAS: LECTURE 1
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1. Algebras

Let k be a field. An *algebra* over k (or k-algebra) is a vector space A, with a bilinear operation

$$\begin{aligned} A \times A &\to A \\ (a, b) &\mapsto a \cdot b. \end{aligned}$$

Recall that bilinearity means that the function is linear over k in each argument, (i.e. preserves sum and multiplication by scalar).

An algebra A is called *associative* if $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in A$. An algebra A is called *commutative* if $a \cdot b = b \cdot a$ for all $a, b \in A$.

Note that to describe an algebra it is enough to determine the product of elements from a vector space basis for A. It is sufficient to check associativity and commutativity of an algebra on basis elements.

Example 1.1. If $K \subset K'$ are fields, then K' is an associative, commutative algebra over K. For instance, $\mathbb{C}$ is an $\mathbb{R}$-algebra.

Example 1.2. The ring of polynomials $k[x_1, \dots, x_n]$ is an associative, commutative k-algebra.

Example 1.3. If V is a vector space, $\mathrm{End}(V)$, the set of (linear) endomorphisms of V is an associative algebra with respect to composition. It is not commutative if $\dim V \geq 2$. If $V = k^n$, then $\mathrm{End}(V)$ is just the algebra of $n \times n$ matrices over k.

Example 1.4. $\mathbb{R}^3$ with respect to the cross product (vector product) $a \times b$ is an algebra. We will see that it is not commutative and not associative.

A vector subspace $B \subseteq A$ is called a *subalgebra* if $a, b \in B$ implies $a \cdot b \in B$. A vector subspace $I \subseteq A$ is called an *ideal* if $a \in A$, $x \in I$ imply $a \cdot x,\ x \cdot a \in I$.

A linear map $f : A \to B$ of k-algebras is called a *homomorphism* if $f(a \cdot b) = f(a) \cdot f(b)$ for each $a, b \in A$. A homomorphism is called an *isomorphism* if it is bijective.

Lemma 1.5. *Let $f : A \to B$ be a homomorphism of algebras. Then the image of f is a subalgebra of B. The kernel of f, $\operatorname{Ker} f = \{a \in A \mid f(a) = 0\}$, is an ideal in A.*

Proof. Let $f(x), f(y) \in \operatorname{Im} f$, and $c_x, c_y \in k$. Then

$$
\begin{aligned}
c_x f(x) + c_y f(y) &= f(c_x x + c_y y) \in \operatorname{Im} f. \\
f(x) f(y) &= f(xy) \in \operatorname{Im} f.
\end{aligned}
$$

Hence, $\operatorname{Im} f$ is a subalgebra of B. Let $x, y \in \ker f$, and $c_x, c_y \in k$. Then

$$f(c_x x + c_y y) = c_x f(x) + c_y f(y) = 0.$$

Let $x \in \ker f$ and $y \in A$. Then

$$f(xy) = f(x)f(y) = 0, \text{ and } f(yx) = f(y)f(x) = 0.$$

Thus, $c_x x + c_y y \in \ker f$ and $xy,\ yx \in \ker f$. Hence, $\ker f$ is an ideal in A. $\square$

Let A be an algebra and I an ideal in A. We define the quotient A/I as follows. As a set this is the quotient of A modulo the equivalence relation

$$a \sim b \iff a - b \in I.$$

So this is the set of equivalence classes of the form $a + I$, where $a \in A$. The structure of a vector space on A/I is given by the formulas

$$(a + I) + (b + I) = (a + b) + I; \ \lambda(a + I) = \lambda a + I.$$

The algebra structure on A/I is given by the formula

$$(a + I) \cdot (b + I) = a \cdot b + I.$$

One has a canonical projection homomorphism $\rho : A \to A/I$, defined by $\rho(a) = a + I$.

Theorem 1.6. *Let $f : A \to B$ be a homomorphism of algebras and let I be an ideal in A. Suppose that $I \subseteq \operatorname{Ker} f$. Then there exists a unique homomorphism $\overline{f} : A/I \to B$ such that $f = \overline{f} \circ \rho$, where $\rho : A \to A/I$ is the canonical homomorphism. Moreover, $\overline{f}$ is surjective iff f is surjective; $\overline{f}$ is injective iff $I = \operatorname{Ker} f$.*

Proof. To satisfy the relation $f = \overline{f} \circ \rho$ one needs that $\overline{f}(a+I) = f(a)$ for all $a \in A$. This map is well-defined since $a+I = b+I$ implies $(a-b) \in I \subseteq \operatorname{Ker} f$, hence $f(a) = f(b)$. Now

$$\overline{f}((a+I)(b+I)) = \overline{f}(ab+I) = f(ab) = f(a)f(b) = \overline{f}(a+I)\overline{f}(b+I),$$

and similarly we obtain

$$\overline{f}((a+I)+(b+I)) = \overline{f}(a+I) + \overline{f}(b+I), \quad \overline{f}(\lambda a + I) = \lambda \overline{f}(a+I).$$

Hence, $\overline{f}$ is a homomorphism. Clearly, the image of $\overline{f}$ coincides with the image of f and $\operatorname{Ker} \overline{f} = \operatorname{Ker} f/I$. This implies the last assertion. □

An element $e \in A$ is called a *unity* if $e \neq 0$ and $ae = ea = a$ for all $a \in A$. Such an element is unique if it exists. We require that any homomorphism between two associative algebras with unity maps unity to unity. A *representation* of an associative k-algebra A with unity is a homomorphism $\rho : A \to \operatorname{End}(V)$, where V is a vector space over k and $\rho(1) = id$.

2. Lie algebras

A *Lie algebra* is a vector space L over a field k, with a bilinear operation $L \times L \to L$, denoted $(x,y) \mapsto [x,y]$ and called the *bracket* or *commutator* of x and y, which satisfies the following axioms, for all $x, y, z \in L$:

(1) $[x,x] = 0$ (anticommutativity),
(2) $[x,[y,z]] = [[x,y],z] + [y,[x,z]]$ (Jacobi identity).

Note that property (1) implies that $[x,y] = -[y,x]$ for all $x, y \in L$. If

$\operatorname{char} k \neq 2$, then this condition is equivalent to (1).

Let A be an associative algebra with bilinear operation denoted $x \cdot y$ for $x, y \in A$. We can define a new operation $[-,-]$,called the bracket of x and y, as follows:

$$[x, y] = x \cdot y - y \cdot x.$$

Then A with the operation $[-,-]$ is a Lie algebra.

Example 2.1. Let V be a finite dimensional vector space over k, and denote by $\operatorname{End}(V)$ the set of linear transformations $V \to V$. Then $\operatorname{End}(V)$ with the bracket operation $[-,-]$ is a Lie algebra, which we write as $\mathfrak{gl}(V)$. It is called the *general linear algebra.* If we choose a basis for V, we may identify $\mathfrak{gl}(V)$ with the set of $n \times n$ matrices over k, and we denote this by $\mathfrak{gl}_n(k)$. The dimension of $\mathfrak{gl}_n(k)$ is n^2.

Any Lie subalgebra of $\mathfrak{gl}(V)$ is called a *linear Lie algebra.* It is a known (but non-trivial) fact that every finite dimensional Lie algebra is isomorphic to some linear Lie algebra (Ado's Theorem).

Example 2.2. Define $\mathfrak{sl}_n = \{x \in \mathfrak{gl}_n \mid \operatorname{Tr}(x) = 0\}$, where $\operatorname{Tr}(x) = \sum x_{ii}$, the sum of the diagonal elements of the matrix x. This is a Lie algebra since for any $x, y \in \mathfrak{sl}_n$,

$$\begin{aligned} &\operatorname{Tr}(c_x x + c_y y) = c_x \operatorname{Tr}(x) + c_y \operatorname{Tr}(y) = 0, \text{ and} \\ &\operatorname{Tr}([x, y]) = \operatorname{Tr}(xy) - \operatorname{Tr}(yx) = 0. \end{aligned}$$

$\mathfrak{sl}_n$ is called the *special linear algebera.* It has dimension $n^2 - 1$.

Example 2.3.

$$\mathfrak{sl}_2(k) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a, b, c \in k \right\}$$

has as a basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then one can check that

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h.$$

Let A be an k-algebra, and let $d : A \to A$ be a linear endomorphism of A (viewing A as a vector space, i.e. $d \in \mathrm{End}_k(A)$). Then $d : A \to A$ is called a *derivation* if the following *Leibniz rule* holds:

$$d(a \cdot b) = d(a) \cdot b + a \cdot d(b).$$

The set of all derivations of A is denoted $\mathrm{Der}(A)$. It is a vector subspace of $\mathrm{End}(A)$. Let $d, d' \in \mathrm{Der}(A)$. The composition dd' is not a derivation, however the bracket, defined as $[d, d'] = dd' - d'd$, is a derivation (i.e $[d, d'] \in \mathrm{Der}(A)$). This can be checked by direct calculation. $\mathrm{Der}(A)$ is a Lie subalgebra of $\mathfrak{gl}(A)$.

Example 2.4. Suppose $L = k$ (i.e. $\dim L = 1$). Let us find all possible Lie brackets on L. Bilinearity and anticommutativity require:

$$[a, b] = [a \cdot 1, b \cdot 1] = ab[1, 1] = 0, \quad \text{for all } a, b \in L.$$

A Lie algebra with a zero bracket is called a *commutative Lie algebra*, or simply *abelian*.

Fix a field k ($\mathrm{char}\, k \neq 2$), and let $L = \langle e_1, \ldots, e_n \rangle$ be an n-dimensional vector space over k. In order to define a bilinear operation, it is enough to define it on e_i:

$$[e_i, e_j] = \sum_{k=1}^{n} c_{ij}^k e_k.$$

The coefficients c_{ij}^k are called the *structure constants* of L. In order for L to be a Lie algebra, it is enough to require that the structure constants satisfy the following equations:

$$c_{ii}^k = 0 = c_{ij}^k + c_{ji}^k,$$
$$\sum_k (c_{ij}^k c_{kl}^m + c_{jl}^k c_{ki}^m + c_{li}^k c_{kj}^m) = 0.$$

This follows from the antisymmetry of the bracket and the Jacobi identity.

Example 2.5. Suppose $\dim L = 2$. Then the Jacobi identity automatically holds. Let us find all possible non-zero brackets on L. Choose a basis $L = \langle e_1, e_2 \rangle$. Then one has

$[e_1, e_1] = [e_2, e_2] = 0,\ [e_1, e_2] = -[e_2, e_1],$ and $[e_1, e_2] = y,$ for some non-zero $y \in L$.

Thus, every bracket in L is proportional y. So we choose y to be on of the generators of L, say $L = \langle x, y \rangle$ and $[x, y] = \lambda y$. Since L is not commutative, $\lambda \neq 0$. Thus, by changing variables $x := x/\lambda$ we obtain:

$$L = \langle x, y \rangle \text{ and } [x, y] = y. \tag{2.1}$$

Therefore, we have shown that there are only two 2-dimensional Lie algebras over k up to isomorphism: a commutative Lie algebra and the one described in (2.1).

Example 2.6. $\mathbb{R}^3$ with respect to the cross product $a \times b$ is a Lie algebra. Recall that the cross product is defined by

$$(a_1, a_2, a_3) \times (b_1, b_2, b_3) = \det \begin{pmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}.$$

It is anticommutative (i.e. $x \times y = -y \times x$) and defined by the formulas

$$e_1 \times e_2 = e_3, \; e_2 \times e_3 = e_1, \; e_3 \times e_1 = e_2.$$

This algebra has no non-trivial ideals. Indeed, let I be a non-zero ideal and $a \in I$ be a non-zero element. Let a, b, c be a basis consisting of mutually orthogonal elements. Then $[a, b]$ is a non-zero multiple of c, and $[a, c]$ is a non-zero multiple of b. Thus $a \in I$ forces $a, b, c \in I$, that is $I = L$.

We call an a Lie algebra L *simple* if $\dim(L) \geq 2$ and L has no non-trivial ideals.