LIE ALGEBRAS: LECTURE 11
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1. Tensor algebra and symmetric algebra

An algebra $\mathcal{A}$ is called *graded* if $\mathcal{A} = \oplus_{i=1}^{\infty} A^i$ such that if $x \in A^j$ and $y \in A^k$ then $xy \in A^{j+k}$. An algebra $\mathcal{A}$ is called *filtered* if there is a sequence of vector subspaces $\{A_i\}_{i=0}^{\infty}$ such that $A_i \subset A_{i+1}$, $A = \cup_{i=0}^{\infty} A_i$ and $A_j A_k \subset A_{j+k}$ for all $j, k \in \mathbb{N}$.

Example 1.1. The polynomial algebra $\mathcal{A} = \mathbb{F}[x_1, \dots, x_n]$ has a natural grading $\mathcal{A} = \oplus_{m=1}^{\infty} A^m$ given by letting A^m be the set of all homogeneous polynomials of degree m. It has a natural filtration $\{A_i\}_{i=0}^{\infty}$ given by letting A_m be the set of all polynomials with degree less than or equal to m.

If $\mathcal{A}$ is a graded algebra $\mathcal{A} = \oplus_{i=1}^{\infty} A^i$, then we can define a filtration $\{A_m\}_{m=0}^{\infty}$ by letting $A_m = A^0 \oplus A^1 \oplus \cdots \oplus A^m$. Conversely, if $\mathcal{A}$ is a filtered algebra $\{A_i\}_{i=0}^{\infty}$, then we can define the *associated graded algebra* $\mathrm{Gr}(A) := \oplus_{i=0}^{\infty} G^i$ where $G^i := A_i / A_{i-1}$ $(A_{-1} = 0)$.

Fix a finite dimensional vector space V over $\mathbb{F}$. Let

$$\begin{array}{ll} T^0 = \mathbb{F} & T^2 = V \otimes V \\ T^1 = V & T^m = V \otimes \cdots \otimes V \ (m \text{ times}). \end{array}$$

Define the *tensor algebra* on V to be $\mathcal{T}(V) := \oplus_{i=0}^{\infty} T^i V$, where multiplication is defined by concatenation, i.e.

$$(x_1 \otimes \cdots \otimes x_j)(y_1 \otimes \cdots \otimes y_k) = x_1 \otimes \cdots \otimes x_j \otimes y_1 \otimes \cdots \otimes y_k.$$

Then $\mathcal{T}(V)$ is an associative graded algebra with identity.

The tensor algebra $\mathcal{T}(V)$ is the universal associative algebra with n generators ($\dim V = n$). This means that given any linear map $\phi : V \to \mathcal{A}$, with $\mathcal{A}$ an associative algebra, there exists a unique homomorphism of algebras

$\psi : \mathcal{T}(V) \to \mathcal{A}$ such that $\psi(1) = 1$ and the following diagram commutes:

$$\begin{array}{ccc} V & \overset{i}{\hookrightarrow} & \mathcal{T}(V) \ , \\ & \underset{\phi}{\searrow} & \downarrow \psi \\ & & \mathcal{A} \end{array}$$

where $i : V \hookrightarrow \mathcal{T}(V)$ is the inclusion map defined by sending V to T^1V.

Next let I be the two sided ideal in $\mathcal{T}(V)$ generated by all $x \otimes y - y \otimes x$ with $x, y \in V$. Define the *symmetric algebra* on V to be $\mathcal{S}(V) := \mathcal{T}(V)/I$. Since I is generated by homogeneous elements $x \otimes y - y \otimes x \in T^2V$, I is a graded ideal. Hence, $\mathcal{S}(V)$ is a graded algebra, so $\mathcal{S}(V) = \oplus_{i=0}^{\infty} S^m$. Since $I \cap T^0 = I \cap T^1 = \varnothing$, we have that $S^0 = \mathbb{F}$ and $S^1 = V$.

The symmetric algebra $\mathcal{S}(V)$ is a commutative algebra, and is canonically isomorphic to the polynomial algebra $\mathbb{F}[x_1, \ldots, x_n]$, where $\{x_1, \ldots, x_n\}$ is a fixed basis of V.

2. Associative algebras

Let $\mathcal{A}$ be an associative algebra. A module V over $\mathcal{A}$ is a vector space V together with an algebra homomorphism $f : \mathcal{A} \to \mathrm{End}(V)$.

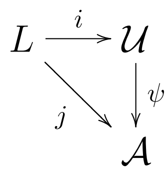
Let $\mathcal{A}$ be an associative algebra. A Lie algebra $\mathcal{A}^{Lie}$ is defined to be the Lie algebra with the same underlying vector space as $\mathcal{A}$ and the bracket defined by $[a, b] : ab - ba$.

Example 2.1. $\mathrm{End}(V)^{Lie} = \mathfrak{gl}(V)$

Let L be a Lie algebra, and let $\mathcal{A}$ be an associative algebra. Then a map $\phi : L \to \mathcal{A}^{Lie}$ is a Lie algebra homomorphism if and only if $\phi : L \to \mathcal{A}$ is a linear map of vector spaces satisfying $\phi([xy]) = \phi(x)\phi(y) - \phi(y)\phi(x)$.

3. Universal enveloping algebra

Let L be a Lie algebra. The *universal enveloping algebra* of L is an associative algebra $\mathcal{U}$ with identity, together with a linear map $i : L \to \mathcal{U}$ satisfying $i([xy]) = i(x)i(y) - i(y)i(x)$ for all $x, y \in L$, and such that for any associative algebra $\mathcal{A}$ with identity and linear map $j : L \to \mathcal{A}$ satisfying $j([xy]) = j(x)j(y) - j(y)j(x)$ there exists a unique homomorphism $\psi : \mathcal{U} \to \mathcal{A}$ such that the following diagram commutes:



We will prove that the universal enveloping algebra exists and is unique, but first we will show that a Lie algebra and its universal enveloping algebra have the same modules.

Let L be a Lie algebra and let $\mathcal{U}$ be the universal enveloping algebra of L with linear map $i : L \to \mathcal{U}$. Suppose V is a module over $\mathcal{U}$ given by the algebra homomorphism $f : \mathcal{U} \to \mathrm{End}(V)$. Let $f' = f \circ i$. Then $f' : L \to \mathrm{End}(V)$ is a linear map satisfying, for all $x, y \in L$,

$$\begin{aligned} f'(x)f'(y) - f'(y)f'(x) - f'([xy]) &= f(i(x)i(y) - i(y)i(x) - i([xy])) \\ &= f(0) = 0. \end{aligned}$$

Hence, $f' : L \to \mathrm{End}(V)^{Lie} = \mathfrak{gl}(V)$ is a Lie algebra homomorphism, and V is an L-module.

Conversely, suppose V is an L-module given by the Lie algebra homomorphism $f' : L \to \mathfrak{gl}(V) = \mathrm{End}(V)^{Lie}$. Then $f' : L \to \mathrm{End}(V)$ is a linear map satisfying $f'(x)f'(y) - f'(y)f'(x) = f'([xy])$ for all $x, y \in L$. By definition of the universal enveloping algebra, there exists a unique algebra homomorphism $f : \mathcal{U} \to \mathrm{End}(V)$ such that $f' = f \circ i$. Hence, V is a $\mathcal{U}$-module.

Lemma 3.1. *Let L be a Lie algebra. The universal enveloping algebra of a Lie algebra is unique (up to isomorphism).*

Proof. Let L be a Lie algebra. Suppose that both $(\mathcal{U}, i)$ and $(\mathcal{U}', i')$ are universal enveloping algebras of L. By the hypothesis, there exist homomorphisms

$\phi : \mathcal{U} \to \mathcal{U}'$ and $\psi : \mathcal{U}' \to \mathcal{U}$ such that the following diagram commutes:

$$\begin{array}{ccc} & & \mathcal{U} \\ & \overset{i}{\nearrow} & \downarrow{\scriptstyle \phi} \\ L & \xrightarrow{i'} & \mathcal{U}' \\ & \underset{i}{\searrow} & \downarrow{\scriptstyle \psi} \\ & & \mathcal{U} \end{array} \;.$$

By definition, there is a unique map $\rho : \mathcal{U} \to \mathcal{U}$ such that the following diagram commutes

$$\begin{array}{ccc} L & \xrightarrow{i} & \mathcal{U} \\ & \underset{i}{\searrow} & \downarrow{\scriptstyle \rho} \\ & & \mathcal{U} \end{array} \;,$$

which is the identity map. Hence, $\psi \circ \phi = \mathrm{id}_{\mathcal{U}}$. Similarly, $\phi \circ \psi = \mathrm{id}_{\mathcal{U}'}$. □

We prove the existence of the universal enveloping algebra for a Lie algebra by construction. Let L be a Lie algebra. Let $\mathcal{T}(L)$ be the tensor algebra of L (where L is viewed as the underlying vector space of the Lie algebra). Let J be the two sided ideal in $\mathcal{T}(L)$ generated by all elements $x \otimes y - y \otimes x - [xy]$ with $x, y \in L$.

Define $\mathcal{U}(L) := \mathcal{T}(L)/J$, and let $\pi : \mathcal{T}(L) \to \mathcal{U}(L)$ be the natural projection map. Define $i : L \to \mathcal{U}(L)$ to be the restriction of the map π to L.

Lemma 3.2. *The associative algebra $\mathcal{U}(L)$ with the linear map $i : L \to \mathcal{U}(L)$ is the universal enveloping algebra of L.*

Proof. Suppose that $\mathcal{A}$ is an associative algebra with a linear map $j : L \to \mathcal{A}$ satisfying $j([xy]) = j(x)j(y) - j(y)j(x)$ for all $x, y \in L$. Since the tensor algebra $\mathcal{T}(L)$ is the universal associative algebra, there exists a unique algebra homomorphism $\phi : \mathcal{T}(L) \to \mathcal{A}$ such that the following diagram commutes:

$$\begin{array}{ccc} L & \overset{i}{\hookrightarrow} & \mathcal{T}(L) \\ & \underset{j}{\searrow} & \downarrow{\scriptstyle \phi} \\ & & \mathcal{A} \end{array} \;.$$

Since $i : L \to \mathcal{T}(L)$ is an injective map and $j([xy]) = j(x)j(y) - j(y)j(x)$ for all $x, y \in L$, we must have that $x \otimes y - y \otimes x - [xy] \in \mathrm{Ker}\ \phi$ for all $x, y \in L$.

Hence, $J \subset \text{Ker } \phi$. Thus ϕ induces a homomorphism $\psi : \mathcal{U}(L) \to \mathcal{A}$ such that $j = \psi \circ i$. □

Example 3.3. If L is abelian, then $\mathcal{U}(L)$ coincides with $\mathcal{S}(L)$.

Now we examine the structure of $\mathcal{U}(L)$. The associative algebra $\mathcal{U}(L)$ is not graded, but it is a filtered algebra, as follows. The tensor algebra $\mathcal{T}(L)$ is graded by $\mathcal{T}(L) = \oplus_{i=0}^{\infty} T^m$, where $T^m = L \otimes \cdots \otimes L$ (m times).

Define a filtration on $\mathcal{T}(L)$ by $T_m = T^0 \oplus T^1 \oplus \cdots \oplus T^m$. Then $T_m \subset T_{m+1}$. Let $\pi : \mathcal{T}(L) \to \mathcal{U}(L)$ be the natural projection map. Define $U_m = \pi(T_m)$. Then $\{U_m\}_{m=0}^{\infty}$ is a filtration of $\mathcal{U}(L)$. The associated graded algebra is $\text{Gr}(\mathcal{U}(L)) = \oplus_{m=0}^{\infty} G^m$ where $G^m = U_m/U_{m-1}$. Define a map

$$\phi^m : T^m \to U_m \to U_m/U_{m-1} = G^m.$$

This map is surjective since $\pi(T_m - T_{m-1}) = U_m - U_{m-1}$. Thus, we have a surjective homomorphism $\phi : \mathcal{T}(L) \to \text{Gr}(\mathcal{U}(L))$ (defined component-wise by $\phi^m : T^m \to G^m$).

Lemma 3.4. $I = \langle x \otimes y - y \otimes x \mid x, y \in L \rangle \subset Ker\ \phi$

Proof. Now $\pi(x \otimes y - y \otimes x) \in U_2$, but also $\pi(x \otimes y - y \otimes x) = \pi([xy]) \in U_1$. Thus $\phi(x \otimes y - y \otimes x) \in U_1/U_1 = 0$. □

Therefore, ϕ induces a surjective map $\psi : \mathcal{S}(L) \to \text{Gr}(\mathcal{U}(L))$.

Theorem 3.5 (Poincare-Birkhoff-Witt Theorem (PBW Theorem))**.** *The homomorphism $\psi : \mathcal{S}(L) \to Gr(\mathcal{U}(L))$ is an isomorphism of algebras.*

Corollary 3.6. *Let $\{x_1, \ldots, x_n\}$ be any ordered basis of L. The the elements $\pi(x_{i_{(1)}} \otimes x_{i_{(2)}} \otimes \cdots \otimes x_{i_{(t)}})$ with $t \in \mathbb{Z}^+$ and $i_{(1)} \leq i_{(2)} \leq \cdots \leq i_{(t)}$ along with 1 form a basis of $\mathcal{U}(L)$.*

Corollary 3.7. *The map $i : L \to \mathcal{U}(L)$ is injective, where i the restriction of the map $\pi : \mathcal{T}(L) \to \mathcal{U}(L)$ to $T^1 = L$.*