LIE ALGEBRAS: LECTURE 14
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1. Finite dimensional modules

Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra. Let

$$\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha\in\Delta}\mathfrak{g}_\alpha)$$

be the corresponding root space decomposition. Choose a set of simple roots Π and corresponding decomposition $\Delta = \Delta^+ \coprod \Delta^-$. Then one has a triangular decomposition, $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, where $\mathfrak{n}^+ = \sum_{\alpha\in\Delta^+}\mathfrak{g}_\alpha,\ \mathfrak{n}^- = \sum_{\alpha\in\Delta^-}\mathfrak{g}_\alpha$. Set

$$Q := \sum_{\alpha\in\Pi}\mathbb{Z}\alpha.$$

Q is called the *root lattice*. Let $Q^+ = \sum_{\alpha\in\Pi}\mathbb{Z}_{\geq 0}\alpha$. Define a partial order on $\mathfrak{h}^*$ by $\lambda \geq \nu$ if $\lambda - \nu \in Q^+$.

Let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be the set of simple roots. Define the *weight lattice* to be

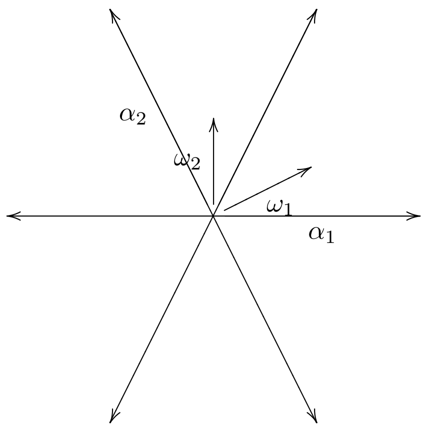
$$\begin{aligned} P(\Pi) :=& \{\lambda \in \mathfrak{h}^* \mid <\lambda, \alpha> \in \mathbb{Z} \text{ for all } \alpha \in \Delta^+\} \\ =& \{\lambda \in \mathfrak{h}^* \mid <\lambda, \alpha> \in \mathbb{Z} \text{ for all } \alpha \in \Pi\}. \end{aligned}$$

Elements of $P(\Pi)$ are called *integral weights*. The *dominant integral weights* are the following:

$$P^+(\Pi) = \{\lambda \in \mathfrak{h}^* \mid <\lambda, \alpha> \in \mathbb{Z}_{\geq 0} \text{ for all } \alpha \in \Pi\}.$$

Note that $\Delta \subset P(\Pi)$. Define the *fundamental weights* $\omega_1, \ldots, \omega_n \in \mathfrak{h}^*$ by the condition $<\omega_i, \alpha_j> = \delta_{ij}$. Then $P^+(\Pi) = \sum_{i=1}^n \mathbb{Z}_{\geq 0}\omega_i$.

Example 1.1. Fundamental weights $\{\omega_1, \omega_2\}$ for $\mathfrak{sl}_3$ with $\Pi = \{\alpha_1, \alpha_2\}$:



Example 1.2. If $\mathfrak{g} = \mathfrak{sl}_n$, then $\Pi = \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n\}$, where by ε_i we mean the image of ε_i in $(\sum_{j=1}^n \mathbb{C}\varepsilon_j)/\mathbb{C}(\sum_{j=1}^n \varepsilon)$. Then $\omega_1 = \varepsilon_1$, $\omega_2 = \varepsilon_1 + \varepsilon_2$, and $\omega_k = \sum_{i=1}^k \varepsilon_i$ for $1 \leq r \leq n-1$.

Let W be the Weyl group of $\mathfrak{g}$. For $\alpha \in \Pi$ and $\lambda \in \mathfrak{h}^*$ define

$$\sigma_\alpha(\lambda) := \lambda - \lambda(h_\alpha)\alpha.$$

Since W is generated by simple reflections, this defines an action of W on $\mathfrak{h}^*$.

Lemma 1.3. *If $w \in W$ and $\lambda \in P(\Pi)$, then $w(\lambda) \in P(\Pi)$.*

Proof. It suffices to prove this for a simple reflection σ_α. Let $\beta \in \Delta^+$, then

$$\langle \sigma_\alpha(\lambda), \beta \rangle = \langle \lambda, \beta \rangle - \lambda(h_\alpha)\langle \alpha, \beta \rangle \in \mathbb{Z}.$$

□

Theorem 1.4. *$V(\lambda)$ is finite dimensional if and only if $\lambda \in P^+(\Pi)$. If $\lambda \in P^+(\Pi)$, then the set of weights of $V(\lambda)$ is permuted by W, with $\dim V_\mu = \dim V_{\sigma(\mu)}$ for $\sigma \in W$.*

We have already proven that if $V(\lambda)$ is finite dimensional then $\lambda \in P^+(\Pi)$. This theorem has the following important corollary.

Corollary 1.5. *The map $\lambda \to V(\lambda)$ induces a one-one correspondence between $P^+(\Pi)$ and the isomorphism classes of finite dimensional irreducible $\mathfrak{g}$-modules.*

Outline of proof for Theorem 1.4. The idea is to show that the set of weights of $V(\lambda)$ is permuted by the Weyl group W, and hence is finite. Suppose now that $\lambda \in P^+(\Pi)$. Fix a maximal vector v^+ of $V = V(\lambda)$ and set $m_i =< \lambda, \alpha_i >$ for $\alpha_i \in \Pi$. For each simple root α_i $(1 \leq i \leq n)$ we have a subalgebra S_i isomorphic to $\mathfrak{sl}(2)$, with generators x_i, y_i, h_i. Denote the representation on V by $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$.

(1) $y_i^{m_i+1} v^+ = 0$. (For $j \neq i$, $x_j y_i^{m_i+1} v^+ = 0$. Set $e_k = \frac{1}{k!} y_i^k v^+$. Then $x_i e_k = (\lambda + 1 - k) e_{k-1}$. So $y_i^{m_i+1} v^+$ is a maximal vector, and hence zero.)

(2) For $1 \leq i \leq n$, V contains a non-zero finite dimensional S_i-module. (The subspace spanned by $y_i^k v^+$ for $1 \leq k \leq m_i$.)

(3) V is the sum of finite dimensional S_i-submodules. (Let T_i be the set of finite dimensional S_i-submodules, and E_i be their sum. One can check that if $F \in T_i$, then $gF \in T_i$ for all $g \in \mathfrak{g}$. Then E_i is a non-zero submodule. Since V is irreducible, $E_i = V$.)

(4) For $1 \leq i \leq n$, $\phi(x_i)$ and $\phi(y_i)$ are locally nilpotent endomorphisms of V (An element $v \in V$ lies in a finite sum of finite dimensional S_i-submodules, and here $\phi(x_i)$ and $\phi(y_i)$ are nilpotent.)

(5) $s_i = \exp \phi(x_i) \exp \phi(-y_i) \exp \phi(x_i)$ is a well-defined automorphism of V. ($\exp \phi(x_i)$ is defined on each finite dimensional S_i-submodule, and agrees on intersections.)

(6) If μ is any weight of V, then $s_i(V_\mu) = V_{\sigma_i(\mu)}$ (where σ_i is the reflection relative to α_i). (In a finite dimensional submodule, s_i is the reflection with respect to the root α_i.)

(7) The set of weights of V is stable under W, and $\dim V_\mu = \dim V_{\sigma(\mu)}$ for $\sigma \in W$. (This follows since W is generated by simple reflections.)

(8) The set of weights of V is finite. We can reflect any weight to the dominant Weyl chamber and its image must be $\leq \lambda$ in order for the

weight space to be non-zero. (The set of dominant integral weights $\mu \leq \lambda$ is finite. Since W is finite, the set of W conjugates of this set is finite. The set of weights of V is included in this set, and hence finite.)

(9) $\dim V$ is finite. (The set of weights is finite by the argument above, and the dimension of each weight space is finite by a lemma from the previous lecture about cyclic modules.)

□

2. Characters

Let $\mathfrak{g}$ be a semisimple Lie algebra and let $\mathfrak{h}$ be a Cartan subalgebra. Let V be a finite dimensional $\mathfrak{g}$-module. A finite dimensional $\mathfrak{g}$-module V is a weight module, i.e. $V = \oplus_{\lambda \in \mathfrak{h}^*} V_\lambda$. (It follows from Weyl's Theorem and a lemma from last lecture that $V = \oplus_{\mu \in \mathfrak{h}^*} V(\mu)^{\oplus n_\mu}$, and each $V(\mu)$ is a weight module by the proposition on cyclic modules.)

Define the *character* of V to be the formal sum:

$$\mathrm{ch}(V) := \sum_{\lambda \in \mathfrak{h}^*} m_\lambda e^\lambda$$

where $m_\lambda = \dim V_\lambda$. The elements e^λ belong to a multiplicative group isomorphic to $\mathfrak{h}^*$ where multiplication is given by $e^\lambda e^\mu = e^{\lambda+\mu}$.

Example 2.1. Let V be the adjoint module of $\mathfrak{sl}_2$. Here $\Delta = \{\alpha, -\alpha\}$. The weight spaces of V have dimension 1. The character of V is

$$e^\alpha + e^0 + e^{-\alpha}.$$

Lemma 2.2. *Let V and W be finite dimensional $\mathfrak{g}$-modules. Then*

(1) $\mathrm{ch}(V \oplus W) = \mathrm{ch}(V) + \mathrm{ch}(W)$

(2) $\mathrm{ch}(V \otimes W) = \mathrm{ch}(V) \cdot \mathrm{ch}(W)$

Proof. The first statement follows from the fact that $(V \oplus W)_\lambda = V_\lambda \oplus W_\lambda$. The second statement follows from exercise 3 on homework 13. □

Define an action of W on $\mathrm{ch}(V)$ by:

$$w\ \mathrm{ch}(V) := \sum_{\lambda \in \mathfrak{h}^*} m_\lambda e^{w(\lambda)}.$$

Proposition 2.3. *Let $\mathfrak{g}$ be a semisimple Lie algebra. If V is a finite dimensional $\mathfrak{g}$-module, then $\mathrm{ch}(V)$ is W invariant: $w\ \mathrm{ch}(V) = \mathrm{ch}(V)$.*

Proof. By Weyl's Theorem, it suffices to prove this statement for irreducible modules since $\mathrm{ch}(V \oplus W) = \mathrm{ch}(V) + \mathrm{ch}(W)$. We already proved this claim for irreducible finite dimensional modules while proving Theorem 1.4. □

Proposition 2.4. *Let $\mathfrak{g}$ be a semsimple Lie algebra. If V and W are finite dimensional $\mathfrak{g}$-modules with $\mathrm{ch}(V) = \mathrm{ch}(W)$, then $V \cong W$.*

Proof. Note that

$$\mathrm{ch}\ V(\lambda) = e^\lambda + \sum_{\mu < \lambda} m_\mu e^\mu.$$

Since V is finite dimensional, we have by Weyl's Theorem that

$$V = \oplus V(\lambda_i)^{\oplus n_i}$$

for some set $\{\lambda_i\}$ with multiplicities n_i. Choose λ_1 to be a maximal element in this set. Then $\dim V_{\lambda_1} = n_{\lambda_1}$ and λ_1 is a maximal weight of V. Since $\mathrm{ch}(V) = \mathrm{ch}(W)$, λ_1 is also a maximal weight of W. Thus, W contains a submodule isomorphic to $V(\lambda_1)^{\oplus n_1}$. This follows from Weyl's Theorem and from the fact that an irreducible finite dimensional module is determine up to isomorphism by its highest weight. Thus,

$$V = L \oplus V(\lambda_1)^{\oplus n_1},\ W = M \oplus V(\lambda_1)^{\oplus n_1}.$$

Thus $\mathrm{ch}(L) = \mathrm{ch}(M)$. Since $\dim L < \dim V$, the result follows by induction on dimension. □

3. Weyl Character Formula

The Weyl character formula allows one to calculate the character of an irreducible $\mathfrak{g}$-module as a function of its highest weight. Before stating the theorem, we will introduce some notation.

Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra. Let Π be a base for Δ. Let

$$\rho = \frac{1}{2} \sum_{\beta \in \Delta^+} \beta.$$

One can show that $\rho(h_\alpha) = 1$ for $\alpha \in \Pi$, so that $\rho \in P^+(\Pi)$.

Theorem 3.1. *For $\lambda \in P^+(\Pi)$ one has that*

$$\mathrm{ch} V(\lambda) = \frac{\sum_{w \in W} (-1)^{l(w)} e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha \in \Delta^+} (1 - e^\alpha)}.$$

For $\lambda = 0$, it follows that

$$1 = \mathrm{ch} V(0) = \frac{\sum_{w \in W} (-1)^{l(w)} e^{w(0+\rho)-\rho}}{\prod_{\alpha \in \Delta^+} (1 - e^\alpha)}.$$

Hence,

$$\prod_{\alpha \in \Delta^+} (1 - e^\alpha) = \sum_{w \in W} (-1)^{l(w)} e^{w(\rho)-\rho}.$$

Corollary 3.2. *For $\lambda \in P^+(\Pi)$ one has that*

$$\mathrm{ch} V(\lambda) = \frac{\sum_{w \in W} (-1)^{l(w)} e^{w(\lambda+\rho)}}{\sum_{w \in W} (-1)^{l(w)} e^{w(\rho)}}.$$