# LIE ALGEBRAS: LECTURE 7
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## 1. $\mathfrak{sl}_n(\mathbb{F})$ is simple

Let $\mathbb{F}$ be an algebraically closed field with characteristic zero. Let $V$ be a finite dimensional vector space. Recall, that $f \in \mathrm{End}(V)$ is said to be diagonalizable if $V$ has a basis of eigenvectors for $f$. This is equivalent to $V = \oplus_{\lambda \in \mathrm{Spec} f} V_\lambda$, where $V_\lambda = \{x \in V \mid f(x) = \lambda x\}$.

**Lemma 1.1.** *Let $V$ be a finite dimensional vector space over $\mathbb{F}$, and let $f_1, \dots, f_k \in End(V)$ be diagonalizable endomorphisms. Then they pairwise commute if and only if they are simultaneously diagonalizable.*

*Proof.* "$\Leftarrow$" If $f_1, \dots, f_k$ are simultaneously diagonalizable, then there exists a basis in which they are diagonal matrices. Since diagonal matrices commute, the $f_i$ pairwise commute.
"$\Rightarrow$" Suppose that $f_1, \dots, f_k$ pairwise commute. We proceed by induction on $k$, with the case $k = 1$ being trivial. Now for any $\lambda \in \mathrm{Spec}\ f_k$, the equalities $f_k f_j = f_j f_k$ imply that $f_j(V_\lambda) \subset V_\lambda$ for $j = 1, \dots, k-1$. Indeed, if $v_\lambda \in V_\lambda$ then $f_k(f_j(v_\lambda)) = f_j(f_k(v_\lambda)) = \lambda(f_j(v_\lambda))$. By induction on $k$, we have that $f_1, \dots, f_{k-1}$ have a common eigenbasis in $V_\lambda$ for each $\lambda \in \mathrm{Spec}\ f_k$. This is a common eigenbasis for $f_1, \dots, f_k$ since $f_k(v_\lambda) = \lambda v_\lambda$ for all $v_\lambda \in V_\lambda$. Since $V = \oplus_{\lambda \in \mathrm{Spec} f_k} V_\lambda$, this implies the existence of a common eigenbasis in $V$. $\square$

If $v$ is an eigenvector for $\vec{f} = (f_1 \dots, f_k)$, then the (generalized) eigenvalue is denoted $\vec{\lambda} = (\lambda_1, \dots, \lambda_k)$, where $\lambda_i$ is the eigenvalue of $f_i$.

**Lemma 1.2.** *Let $V$ be a finite dimensional vector space over $\mathbb{F}$, and let $f_1, \dots, f_k \in End(V)$ be diagonalizable pair-wise commuting endomorphisms. Suppose that $W$ is a subspace of $V$ such that $f_i(W) \subset W$ for $i = 1, \dots, k$. Then*

$$W = \oplus_{\vec{\lambda}} W_{\vec{\lambda}}, \text{ where } W_{\vec{\lambda}} = W \cap V_{\vec{\lambda}}.$$

*Proof.* We proceed by induction on $k$. The case $k = 1$ was proven last lecture. Write

$$V = \oplus_{\mu \in \mathrm{Spec} f_k} V_\mu.$$

Then by the lemma from last lecture, we have that

$$W = \oplus_\mu W_\mu, \text{ where } W_\mu := W \cap V_\mu.$$

As shown before, each eigenspace $V_\mu$ is invariant with respect to $f_1 \ldots, f_{k-1}$. Thus,

$$V_\mu = \oplus_{\vec{\lambda}: \lambda_k = \mu} V_\lambda.$$

By applying induction to $f_1 \ldots, f_{k-1}$ on the vector space $V_\mu$, we have that

$$W_\mu = \oplus_{\vec{\lambda}: \lambda_k = \mu} W_{\vec{\lambda}}, \text{ where } W_{\vec{\lambda}} = W \cap V_{\vec{\lambda}} \text{ with } \lambda_k = \mu.$$

Therefore,

$$W = \oplus_{\vec{\lambda}} W_{\vec{\lambda}}.$$

□

Let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{F})$. Let $\mathfrak{h} = \sum \mathbb{F} E_{ii}$ be the set of diagonal matrices. Then $\mathfrak{h}$ consists of ad-diagonalizable elements (i.e. for $x \in \mathfrak{h}$, $\mathrm{ad}\, x$ is diagonal with respect to the natural basis of $\mathfrak{gl}_n$), and $[\mathrm{ad}\ x, \mathrm{ad}\ y] = \mathrm{ad}\ [x, y] = 0$ for any $x, y \in \mathfrak{h}$. Set $\phi_i = \mathrm{ad}\ E_{ii} \in \mathrm{End}(\mathfrak{gl}_n)$. Then the eigenspaces of $\{\phi_i\}_{i=1}^n$ are: $\mathfrak{h}$ (with generalized eigenvalue $\vec{0}$) and $\mathbb{F} E_{ij}$ with $i \neq j$.

We can view each generalized eigenvalue $(\lambda_1, \ldots, \lambda_n)$ as an element of $\mathfrak{h}^*$ by defining $\alpha \in \mathfrak{h}^*$ to be $\alpha(E_{ii}) := \lambda_i$. Then write

$$\mathfrak{gl}_n = \oplus_{\alpha \in S} \mathfrak{g}_\alpha$$

where $S \subset \mathfrak{h}^*$. We will explicitly describe the set S.

Choose a basis $\varepsilon_1, \ldots, \varepsilon_n$ in $\mathfrak{h}^*$ which is dual to the basis $\{E_{ii}\}_{i=1}^n$ in $\mathfrak{h}$, so that $\varepsilon_j(E_{ii}) = \delta_{ij}$. Then

$$S = \{0, \varepsilon_i - \varepsilon_j \mid i \neq j\}$$

and

$$\mathfrak{g}_0 = \mathfrak{h}, \qquad \mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \mathbb{F} E_{ij}.$$

Note that $\alpha \in S$ implies $-\alpha \in S$, and that $\mathfrak{g}_\alpha$ are one dimensional for $\alpha \neq 0$.

**Lemma 1.3.** *The Lie algebra $\mathfrak{sl}_n(\mathbb{F})$ is simple.*

*Proof.* Since $\mathfrak{gl}_n = \mathfrak{sl}_n \times \mathbb{F}\mathrm{id}$, it suffices to show the $\mathfrak{sl}_n$ and $\mathbb{F}\mathrm{id}$ (scalar matrices) are the only non-trivial ideals in $\mathfrak{gl}_n$. Let $I$ be an ideal in $\mathfrak{gl}_n$ with $I \neq 0, \mathbb{F}\mathrm{id}$. We will show that $\mathfrak{sl}_n \subset I$. Since the ideal $I$ is ad $\mathfrak{h}$-invariant, by Lemma 1.2, either $I$ contains $E_{ij}$ for some $i \neq j$ or $I \cap \mathfrak{h} \neq 0, \mathbb{F}\mathrm{id}$. In the second case, take $x \in I \cap \mathfrak{h}$, $x \notin \mathbb{F}\mathrm{id}$. Then $x = \sum a_i E_{ii}$ where $a_i \neq a_j$ for some $i, j$. Then $I$ contains $E_{ij}$, since $[x, E_{ij}] = (a_i - a_j)E_{ij} \neq 0$. Thus in both cases $I$ contains $E_{ij}$ for some $i \neq j$. Now if $k \neq i$ then $E_{ik} = [E_{ij}, E_{jk}] \in I$. Thus for all $k \neq i$, $E_{ii} - E_{kk} = [E_{ik}, E_{ki}] \in I$. The elements $E_{ii} - E_{kk}$ span $\mathfrak{h} \cap \mathfrak{sl}_n = \{\sum b_r E_{rr} \mid \sum b_r = 0\}$. Finally, for any $1 \leq r, s \leq n$ with $r \neq s$ we have $2E_{rs} = [E_{rr} - E_{ss}, E_{rs}] \in I$. Thus, $E_{rs} \in I$ for $r \neq s$. Hence, $\mathfrak{sl}_n \subset I$. $\square$

## 2. Cartan subalgebra and the root space decomposition

Let $\mathfrak{g}$ be a Lie algebra. A *Cartan subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ is a nilpotent subalgebra which equals its normalizer (i.e. $N(\mathfrak{h}) = \mathfrak{h}$, $N(\mathfrak{h}) = \{x \in \mathfrak{g} \mid [x, \mathfrak{h}] \subset \mathfrak{h}\}$). The existence of a Cartan subalgebra for an algebra $\mathfrak{g}$ is proven in Humphreys Section 15. Moreover, it is shown in Section 16 that all Cartan subalgebras are conjugate. This means that if $\mathfrak{h}, \mathfrak{h}'$ are Cartan subalgebras, then there exists an (inner) automorphism $\psi : \mathfrak{g} \to \mathfrak{g}$ with $\psi(\mathfrak{h}) = \mathfrak{h}'$.

An element $x \in \mathfrak{g}$ is called *semisimple* if ad $x$ is a semisimple endomorphism, i.e. diagonalizable when $\mathbb{F}$ is algebraically closed.

**Theorem 2.1.** *Let $\mathfrak{h}$ be a Cartan subalgebra of a semisimple Lie algebra $\mathfrak{g}$. Then*

(1) *$\mathfrak{h}$ is abelian;*
(2) *the centralizer of $\mathfrak{h}$ is $\mathfrak{h}$;*
(3) *every element of $\mathfrak{h}$ is semisimple.*

*Remark* 2.2. In fact, if $\mathfrak{g}$ is semisimple then the Cartan subalgebras of $\mathfrak{g}$ are precisely the maximal abelian subalgebras which consist of semisimple elements.

Let $\mathfrak{g}$ be a semisimple Lie algebra, and let $\mathfrak{h}$ be a Cartan subalgebra with basis $\{h_1, \ldots, h_n\}$. By the theorem, the endomorphisms ad $h_i$ are diagonalizable and they pairwise commute, hence they are simultaneously diagonalizable. Thus, we have proven that $\mathfrak{g}$ decomposes into a direct sum of eigenspaces $\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for } h \in \mathfrak{h}\}$ with $\alpha \in \mathfrak{h}^*$. Note that $\mathfrak{g}_0 = \mathfrak{h}$. An element $\alpha \in \mathfrak{h}^*$ is called a *root* if $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$. Let $\Delta$ denote the set of roots. Hence, we have a *root space decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha \in \Delta} \mathfrak{g}_\alpha).$$

**Lemma 2.3.** *Let $\mathfrak{g}$ be a semisimple Lie algebra, and let $\mathfrak{h}$ be a Cartan subalgebra. For all $\alpha, \beta \in \mathfrak{h}^*$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$. If $x \in \mathfrak{g}_\alpha$ with $\alpha \neq 0$, then ad $x$ is nilpotent. If $\alpha, \beta \in \mathfrak{h}^*$, and $\alpha + \beta \neq 0$, then $\mathfrak{g}_\alpha$ is orthogonal to $\mathfrak{g}_\beta$ relative to the Killing form of $\mathfrak{g}$.*

*Proof.* Let $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$. Then by the Jacobi identity:

$$\begin{aligned} [h, [x, y]] &= [[h, x], y] + [x, [h, y]] \\ &= \alpha(h)[x, y] + \beta(h)[x, y] \\ &= (\alpha + \beta)(h)[x, y]. \end{aligned}$$

For the second statement, let $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$. Then $(\text{ad } x)^n(y) \in \mathfrak{g}_{n\alpha+\beta}$. The nonzero vectors of the form $(\text{ad } x)^n(y)$ with $n \in \mathbb{N}$ are linearly independent. Since $\mathfrak{g}$ finite dimensional, there exists an $n$ such that $(\text{ad } x)^n(y) = 0$. Since ad $x$ is nilpotent on a basis for $\mathfrak{g}$ and $\mathfrak{g}$ is finite dimensional, there exists some $N \in \mathbb{N}$ such that $(\text{ad } x)^N = 0$ on $\mathfrak{g}$. Finally, let $h \in \mathfrak{h}$ such that $(\alpha + \beta)(h) \neq 0$. Then for $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$,

$$\begin{aligned} \alpha(h)\kappa(x, y) &= \kappa([h, x], y) = -\kappa([x, h], y) \\ &= -\kappa(x, [h, y]) = -\beta(h)\kappa(x, y). \end{aligned}$$

Hence, $(\alpha + \beta)(h)\kappa(x, y) = 0$ implies that $\kappa(x, y) = 0$. □

**Corollary 2.4.** *If $\alpha \in \Delta$, then $-\alpha \in \Delta$.*

**Corollary 2.5.** *The restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{h}$ is non-degenerate.*

We may identify $\mathfrak{h}$ with $\mathfrak{h}^*$ as follows. To each $\phi \in \mathfrak{h}^*$ there corresponds a unique $t_\phi \in \mathfrak{h}$ such that $\phi(h) = \kappa(t_\phi, h)$ for all $h \in \mathfrak{h}$.

**Proposition 2.6.** *Let $\mathfrak{g}$ be a semisimple Lie algebra, and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Let $\Delta$ be the corresponding set of roots. Then:*

(1) *$\Delta$ is finite, spans $\mathfrak{h}^*$ and $0 \notin \Delta$.*

(2) *If $\alpha \in \Delta$, $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$, then $[x, y] = \kappa(x, y)t_\alpha$.*
*Hence, $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is one dimensional.*

(3) *$\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$.*

(4) *If $\alpha \in \Delta$ and $x_\alpha$ is any non-zero element of $\mathfrak{g}_\alpha$, then there exists $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $x_\alpha, y_\alpha, h_\alpha = [x_\alpha, y_\alpha]$ span a three dimensional simple subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2)$.*

(5) *$h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$; $h_\alpha = -h_{-\alpha}$.*

*Proof.* (1) It is clear that $\Delta$ is finite, and $0 \notin \Delta$. Suppose $\Delta$ does not span $\mathfrak{h}^*$. Then there exists an $h \in \mathfrak{h}$ such that $\alpha(h) = 0$ for all $\alpha \in \Delta$. Then $[h, \mathfrak{g}_\alpha] = 0$ for all $\alpha \in \Delta$, which implies $h \in Z(\mathfrak{g})$. Contradiction.

(2) For all $h \in \mathfrak{h}$ $\kappa(h, [x, y]) = \kappa([h, x], y) = \alpha(h)\kappa(x, y) = \kappa(t_\alpha, h)\kappa(x, y) = \kappa(\kappa(x, y)t_\alpha, h) = \kappa(h, \kappa(x, y)t_\alpha)$. Since $\kappa$ is non-degenerate, $[x, y] = \kappa(x, y)t_\alpha$.

(3) Suppose $\alpha(t_\alpha) = 0$. Choose $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$ such that $\kappa(x, y) = 1$. Then $x, y, t_\alpha$ form a solvable subalgebra, since $[t_\alpha, x] = 0$ and $[t_\alpha, y] = 0$. Now $S \cong \operatorname{ad}_\mathfrak{g} S \subset \mathfrak{gl}(\mathfrak{g})$. Hence, for any $s \in [S, S]$ we have that $\operatorname{ad}_\mathfrak{g} s$ is nilpotent. Since $\operatorname{ad}_\mathfrak{g} t_\alpha$ is also semisimple, this implies $\operatorname{ad}_\mathfrak{g} t_\alpha = 0$ and $t_\alpha \in Z(\mathfrak{g})$. Contradiction.

(4) Choose $x_\alpha \in \mathfrak{g}_\alpha$, $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $\kappa(x_\alpha, y_\alpha) = \frac{2}{\kappa(t_\alpha, t_\alpha)}$. Let $h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$. Then $x_\alpha, y_\alpha, h_\alpha$ is isomorphic to $\mathfrak{sl}(2)$.

(5) Since $t_\alpha$ is defined by $\kappa(t_\alpha, h) = \alpha(h)$ we have that

$$\kappa(-t_\alpha, h) = -\alpha(h) = \kappa(t_{-\alpha}, h).$$

Since the Killing form is nondegenerate, we have that $t_\alpha = -t_{-\alpha}$.

□