LIE ALGEBRAS: LECTURE 9
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1. Bases

A subset Π of Δ is called a *base* if Π is a basis for E and each root $\beta \in \Delta$ can be (uniquely) written as $\beta = \sum_{\alpha\in\Pi} k_\alpha \alpha$ with integral coefficients k_α, and all non-negative or all non-positive. Elements of Π are called *simple roots.*

We define the *height* of β to be $\mathrm{ht}(\beta) = \sum_{\alpha\in\Pi} k_\alpha$. If $k_\alpha > 0$ (resp. $k_\alpha < 0$) we call β *positive* (resp. *negative*) and write $\beta > 0$ (resp. $\beta < 0$). We denote by Δ^+ (resp. Δ^-) the collection of positive (resp. negative) roots. This defines a partial order on E by $\lambda > \mu$ if and only if $\lambda - \mu$ is a sum of positive roots. Note that $\Delta^- = -\Delta^+$.

Now we prove the existence of a base for a root system Δ. For each $\alpha \in \Delta$, let $P_\alpha = \{v \in E \mid (\alpha, v) = 0\}$. Choose a vector $\nu \in E - \cup_{\alpha\in\Delta} P_\alpha$. Then for all $\alpha \in \Delta$ we have that $\alpha \notin P_\nu$. Let $\Delta^+(\nu) = \{\alpha \in \Delta \mid (\nu, \alpha) > 0\}$. Then $\Delta = \Delta^+(\nu) \cup (-\Delta^+(\nu))$.

Call $\alpha \in \Delta^+(\nu)$ *decomposable* if $\alpha = \beta_1 + \beta_2$ for some $\beta_1, \beta_2 \in \Delta^+(\nu)$, and *indecomposable* otherwise. Let $\Pi(\nu)$ be the set of indecomposable elements.

Lemma 1.1. *Each element of $\Delta^+(\nu)$ is a linear combination, with non-negative integer coefficients, of elements of $\Pi(\nu)$.*

Proof. Let I be the set of $\alpha \in \Delta^+(\nu)$ which cannon be written as such. If I is non-empty, choose $\alpha \in I$ with (ν, α) minimal. The element α is decomposable, since otherwise it would belong to $\Pi(\nu)$. Write $\alpha = \beta_1 + \beta_2$ with $\beta_1, \beta_2 \in \Delta^+(\nu)$. Then $(\nu, \alpha) = (\nu, \beta_1) + (\nu, \beta_2)$ and $0 < (\nu, \beta_1), (\nu, \beta_2) < (\nu, \alpha)$, which implies $\beta_1, \beta_2 \notin I$. This implies $\alpha \notin I$, which is a contradiction. $\square$

We know that the set $\Pi(\nu)$ spans E, since Δ spans E. To conclude that $\Pi(\nu)$ is a base for Δ it only remains to be shown that the set is linearly independent. This will follows from the next two lemmas.

Lemma 1.2. *If $\alpha, \beta \in \Pi(\nu)$ with $\alpha \neq \beta$, then $(\alpha, \beta) \leq 0$.*

Proof. Otherwise $\mu = \alpha - \beta$ would be a root, by a lemma from last lecture. If $\mu \in \Delta^+(\nu)$, then $\alpha = \mu + \beta$ would be decomposable. If $-\mu \in \Delta^+(\nu)$, then $\beta = \alpha - \mu$ would be decomposable. $\square$

Lemma 1.3. *Let $\nu \in E$ and suppose that $S \subset E$ such that: $(\nu, \alpha) > 0$ for all $\alpha \in S$, and $(\alpha, \beta) \leq 0$ for all $\alpha \neq \beta$ in S. Then the elements of S are linearly independent.*

Proof. Each relation between elements of S can be written in the form

$$\sum a_\alpha \alpha = \sum b_\beta \beta$$

where the coefficients a_α and b_β are non-negative, and where α and β range over disjoint finite subsets of S. Let $\lambda = \sum a_\alpha \alpha$. Then

$$(\lambda, \lambda) = \sum a_\alpha b_\beta (\alpha, \beta)$$

implying that $(\lambda, \lambda) \leq 0$. Hence, $\lambda = 0$ since $(\cdot, \cdot)$ is positive definite. But we also have that

$$0 = (\nu, \lambda) = \sum a_\alpha (\nu, \alpha)$$

which implies that all the coefficients a_α must be zero. Similarly, the coefficients b_β must be zero. $\square$

Lemma 1.4. *Each base Π for Δ has the form $\Pi(\nu)$ for some $\nu \in E$.*

Proof. Choose $\nu \in E$ such that $(\nu, \alpha) > 0$ for $\alpha \in \Pi$. This is possible since the intersection of "positive" open half-spaces associated with a basis is non-empty. Then $\Delta^+ \subset \Delta^+(\nu)$ and $\Delta^- \subset \Delta^-(\nu)$. Hence, equality holds and $\Pi = \Pi(\nu)$. $\square$

The hyperplanes P_α with $\alpha \in \Delta$ partition the vector space E into finitely many regions. The connected components of $E - \cup_{\alpha \in \Delta} P_\alpha$ are called *Weyl chambers*. Now ν and μ belong to the same Weyl chamber if and only if $\Delta^+(\nu) = \Delta^+(\mu)$, equivalently $\Pi(\nu) = \Pi(\mu)$. Hence, there is a one-one correspondence between Weyl chambers and bases.

2. The Weyl group

The *Weyl group*, denoted by W, is the subgroup of $GL(E)$ generated by the reflections σ_α for $\alpha \in \Delta$. Now W permutes the finite set Δ. Since Δ spans E, we can identify W with a subgroup of the symmetric group on the set Δ. Hence, W is finite.

Let Π be a fixed base for Δ.

Lemma 2.1. *If β is a simple root, then σ_β permutes the set $\Delta^+ \setminus \{\beta\}$.*

Proof. Let $\mu \in \Delta^+ \setminus \{\beta\}$. Then $\mu = \sum_{\alpha\in\Pi} k_\alpha \alpha$ with $k_\alpha \in \mathbb{Z}_{\geq 0}$. Since μ is not proportional to β, there is some simple root $\alpha' \neq \beta$ such that $k_{\alpha'} > 0$. Now

$$\sigma_\beta(\mu) = \mu - \langle \mu, \beta\rangle\beta = \sum_{\alpha\in\Pi} k_\alpha\alpha - \langle \mu, \beta\rangle\beta,$$

so the coefficient of α' in $\sigma_\beta(\mu)$ is also equal to $k_{\alpha'}$ which is positive. Hence, $\sigma_\beta(\mu) \in \Delta^+$. □

Lemma 2.2. *Let $\alpha_1, \ldots, \alpha_t \in \Pi$, not necessarily distinct. Let $\sigma_i = \sigma_{\alpha_i}$. If $\sigma_1 \cdots \sigma_t(\alpha_t)$ is positive, then for some index $1 \leq s < t$, we have that $\sigma_1 \cdots \sigma_t = \sigma_1 \cdots \sigma_{s-1}\sigma_{s+1} \cdots \sigma_{t-1}$.*

Proof. Now $\sigma_t(\alpha_t) = -\alpha_t$, so $\sigma_1 \cdots \sigma_t(\alpha_t) = -\sigma_1 \cdots \sigma_{t-1}(\alpha_t)$. Write

$$\beta_i = \sigma_{i+1} \cdots \sigma_{t-1}(\alpha_t).$$

Since $\beta_0 < 0$ and $\beta_{t-1} > 0$, we can find the smallest index s for which $\beta_s > 0$. Then $\sigma_s(\beta_s) = \beta_{s-1} < 0$. By the previous lemma, $\beta_s = \alpha_s$, hence $\alpha_s = \beta_s = \sigma_{s+1} \cdots \sigma_{t-1}(\alpha_t)$. Recall that $\sigma_{\sigma(\alpha)} = \sigma\sigma_\alpha\sigma^{-1}$. Therefore, $\sigma_s = (\sigma_{s+1} \cdots \sigma_{t-1})\sigma_t(\sigma_{t-1} \cdots \sigma_{s+1})$, and substitution yields the result. □

Let W' be the subgroup of W generated by reflections σ_α with $\alpha \in \Pi$. We will show that $W = W'$.

Lemma 2.3. *If $\beta \in \Delta$ then there exists $\sigma \in W'$ such that $\sigma(\beta) \in \Pi$. Hence, $\Delta = W(\Pi)$.*

Proof. Let $\beta \in \Delta$, we will show that there exists an $\alpha \in \Pi$ and $\sigma \in W'$ such that $\sigma(\beta) = \alpha$. It suffices to prove this for $\beta \in \Delta^+$, since if $\sigma(\beta) = \alpha$

with $\alpha \in \Pi$, $\sigma \in W'$, then $\sigma_\alpha \sigma(-\beta) = \alpha$ with $\sigma_\alpha \sigma \in W'$. Now we assume $\beta \in \Delta^+$, and we prove the lemma by induction on the height of β. Recall that if $\beta = \sum_{\alpha \in \Pi} k_\alpha \alpha$, then $\mathrm{ht}(\beta) = \sum_{\alpha \in \Pi} k_\alpha$. If $\mathrm{ht}(\beta) = 1$, then β is simple and we are done.

So suppose now that $\mathrm{ht}(\beta) \geq 2$. Then by Lemma 1.3, there is some $\mu \in \Pi$ such that $(\beta, \mu) > 0$, since otherwise $\Pi \cup \{\beta\}$ would be a linearly independent set in E. Then $\langle \beta, \mu \rangle = \frac{2(\beta,\mu)}{(\mu,\mu)} \in \mathbb{Z}_{\geq 1}$. Then $\sigma_\mu(\beta) = \beta - \langle \beta, \mu \rangle \mu$ and $\mathrm{ht}(\sigma_\mu(\beta)) = \mathrm{ht}(\beta) - \langle \beta, \mu \rangle < \mathrm{ht}(\beta)$. By induction hypothesis, there exists a $\sigma \in W'$ and $\alpha \in \Pi$ such that $\sigma(\sigma_\mu(\beta)) = \alpha$. Since $\sigma\sigma_\mu \in W'$, we are done. $\square$

Lemma 2.4. *The Weyl group is generated by reflections of simple roots, ie. by σ_α for $\alpha \in \Pi$.*

Proof. Clearly, W' is a subgroup of W. For equality, we need to show that $\sigma_\beta \in W'$ for each $\beta \in \Delta$. Let $\beta \in \Delta$ and find $\sigma \in W'$ such that $\sigma(\beta) = \alpha \in \Pi$. Then $\sigma_\alpha = \sigma_{\sigma(\beta)} = \sigma\sigma_\beta\sigma^{-1}$. Hence, $\sigma_\beta = \sigma^{-1}\sigma_\alpha\sigma \in W'$. Therefore, $W' = W$. $\square$

If $\sigma \in W$ is written as $\sigma_{\alpha_1} \cdots \sigma_{\alpha_t}$ ($\alpha_i \in \Pi$) with t minimal, then the expression is called *reduced* and t is called the *length* of σ, denoted $l(\sigma)$.

Proposition 2.5. *If Π' is another base of Δ, then there exists $\sigma \in W$ such that $\sigma(\Pi') = \Pi$. If $\sigma \in W$ such that $\sigma(\Pi) = \Pi$, then $\sigma = 1$. Thus, the Weyl group acts simply transitively on the set of bases of Δ.*

Proof. This statement is equivalent to the assertion that the Weyl group acts simply transitively on the set of decompositions of Δ into $\Delta = \Delta^+ \sqcup \Delta^-$. First, we will show that W acts transitively on this set. Let $\Delta = \Delta^+ \sqcup \Delta^-$ be a decomposition of Δ, and $\Delta = \Delta'^+ \sqcup \Delta'^-$ be another decomposition. We will show that there exists some $\sigma \in W$ such that $\sigma(\Delta'^+) = \Delta^+$.

We prove this by induction on the number of roots in Δ^+ which are not contained in Δ'^+. This number is finite, since Δ is a finite set. If this number is zero, then $\Delta^+ \subset \Delta'^+$ and $\Delta^- = -\Delta^+ \subset -\Delta'^+ = \Delta'^-$ implies that

$\Delta^+ = \Delta'^+$. If this number is non-zero, then there exists a simple root $\alpha \in \Delta^+$ which is not in Δ'^+. We claim that the number of roots in $\Delta^+ \cap \sigma_\alpha(\Delta'^+)$ is strictly greater than the number of roots in $\Delta^+ \cap \Delta'^+$. Indeed, since σ_α permutes the set $\Delta^+ \setminus \{\alpha\}$ and $(\Delta'^+ \cap \Delta^+) \subset \Delta^+ \setminus \{\alpha\}$ we have that

$$\sigma_\alpha(\Delta'^+) \cap \Delta^+ \supset \sigma_\alpha(\Delta'^+ \cap \Delta^+) \sqcup \{\alpha\} = \sigma_\alpha((\Delta'^+ \cap \Delta^+) \sqcup \{-\alpha\}).$$

Hence, $|\Delta^+ \cap \sigma_\alpha(\Delta'^+)| > |\Delta^+ \cap \Delta'^+|$. The induction hypothesis applies to $\sigma_\alpha(\Delta'^+)$ to give the existence of a $\sigma \in W$ such that $\sigma\sigma_\alpha(\Delta'^+) = \Delta^+$. Thus, the Weyl group acts transitively.

Suppose $\sigma \in W$ such that $\sigma(\Pi) = \Pi$, but $\sigma \neq 1$. Let $\sigma = \sigma_1 \dots \sigma_t$ be a reduced expression for σ. But then by a Lemma 2.2, $\sigma(\alpha_t) \in \Delta^-$, which is a contradiction. □

Corollary 2.6. *The Weyl group acts simply transitively on the Weyl chambers.*