Dimension-free variance bounds for polynomials

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Joint work with Itay Glazer (Northwestern)
Let \( \{X_i\}_{i=1}^k \) be i.i.d. copies of an isotropic random vector \( X \sim \mu \) in \( \mathbb{R}^n \). And consider

\[
W := \frac{1}{\sqrt{k}} \sum_{i=1}^k \left( X_i \otimes d - \mathbb{E} [X_i \otimes d] \right).
\]

Keeping \( n \) and \( d \) fixed the \( W \) converges to a Gaussian vector. What happens when we allow \( n \) (and \( d \)) to approach infinity?
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Keeping $n$ and $d$ fixed the $W$ converges to a Gaussian vector. What happens when we allow $n$ (and $d$) to approach infinity?
Some motivation to understand the asymptotic normality of $W$:

1. **Empirical moment tensor estimation.**

2. Related to random geometric graphs, when $d = 2$.

\[
XX^T = \sum_{i=1}^{k} X_i \otimes X_i
\]

where $X$ is a matrix with columns given by $X_i$.

3. CLT for neural networks, when $d \geq 2$. For fixed $y \in \mathbb{R}^n$,

\[
\frac{1}{\sqrt{k}} \sum_{i=1}^{k} \langle X_i, y \rangle^p = \frac{1}{\sqrt{k}} \sum_{i=1}^{k} \langle X_i^\otimes p, y^\otimes p \rangle = \langle \frac{1}{\sqrt{k}} \sum_{\ell=1}^{k} X_i^\otimes p, y^\otimes p \rangle.
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Known results

When $n^{2d-1} \ll k$, $W$ is asymptotically normal.

- Bubeck, Ding, Eldan, Rácz 15’ and Jiang, Li 15’ - $d = 2$, standard Gaussian.
- Bubeck, Ganguly 15’ - $d = 2$, log-concave product measures.
- Fang, Koike 20’ - $d = 2$, product measures.
- Nourdin, Zheng 18’- $d \geq 2$, standard Gaussian.
- M. 20’ - $d \geq 2$, unconditional strongly log-concave measures.
- M., Shenfeld 21’ - $d \geq 2$, unconditional log-concave measures.

$\mu$ is log-concave if $-\log\left(\frac{d\mu}{dx}\right)$ convex.

$\mu$ is unconditional if $\frac{d\mu}{dx}(x_1, \ldots, x_n) = \frac{d\mu}{dx}(\pm x_1, \ldots, \pm x_n)$. 
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Important caveat:
Instead of considering the full tensor $W$, the results apply to its marginal on the subspace of principal (multi-linear) tensors:

$$\text{span} \{ e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} | i_1 < i_2 < \cdots < i_k \}.$$ 

Reason: If $X = (X_1, \ldots, X_n)$ is unconditional, the covariance matrix on the principal subspace is diagonal:

$$E[(X_{i_1} \cdots X_{i_k})(X_{j_1} \cdots X_{j_k})] = 0,$$

whenever $(i_1, \ldots, i_k) \neq (j_1, \ldots, j_k)$. 
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Remark

To control convergence rate of the CLT, one needs to understand \( \lambda_{\text{min}}(\text{Cov}(X \otimes d)) \) and \( \lambda_{\text{max}}(\text{Cov}(X \otimes d)) \).

To rephrase, let \( f : \mathbb{R}^n \to \mathbb{R} \), be a degree \( d \) homogeneous polynomial \( f(x) = \sum_I v_I x^I \), where

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I \in [n]^d \text{ and } x^I = \prod_{i=1}^d x_{I_i}.
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So, \( \langle X \otimes d, f \rangle = \sum_{|I|=d} v_I X^I = f(X) \), and

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\lambda_{\text{min}}(\text{Cov}(X \otimes d)) = \inf_{f: \sum v_i^2 = 1} \text{Var}(f(X)).
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$$\lambda_{\text{min}} \left( \text{Cov}(X \otimes^d) \right) = \inf_{f : \sum v_i^2 = 1} \text{Var}(f(X)).$$
Lemma

Let $G$ be a standard Gaussian in $\mathbb{R}^n$, and let $f(x) = \sum_{i} v_i x_i$ with $\sum v_i^2 = 1$. Then, $\text{Var}(f(G)) \geq \frac{1}{d!}$.

Proof.

Gaussian integration by parts:

$$
\text{Var}(f(G)) = \sum_{m=1}^{\infty} \frac{\|E[\nabla^m f(G)]\|^2}{m!} \geq \frac{\|E[\nabla^d f(G)]\|^2}{d!}.
$$

But,

$$
\frac{d}{dx^I} x^J = I! \delta_{IJ} \quad \Rightarrow \quad \frac{d}{dx^I} f = I! v_I.
$$

So,

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\|E[\nabla^d f(G)]\|^2 = \sum (I!)^2 v_I^2 \geq 1.
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**Lemma**

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**Proof.**

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The previous proof is very Gaussian.

**Question**

Which isotropic random vectors satisfy,

\[ \text{Var}(f(X)) \geq C_d, \]

for any \( d \)-homogeneous polynomial with \( \sum v_i^2 = 1 \)?

Specific cases of interest:

1. Product measures.
2. Log-concave measures.
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Which isotropic random vectors satisfy,

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**Specific cases of interest:**

1. Product measures.
2. Log-concave measures.
The celebrated Carbery-Wright inequality connects between log-concave measures and level sets of polynomials.

**Lemma (Carbery-Wright’s inequality)**

Let $X$ be a log-concave vector in $\mathbb{R}^m$, then for any polynomial $f$ of degree $d$, $t \in \mathbb{R}$ and $\varepsilon$.

$$\mathbb{P}(|f(X) - t| < \varepsilon) \lesssim \left( \frac{\varepsilon}{\sqrt{\mathbb{E}[f(X)^2]}} \right)^{\frac{1}{d}}$$

Problems:

- Need to show that $\mathbb{E}[f^2(X)]$ is comparable to $\sum v_i^2$.
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A more fundamental problem is that Carbery-Wright is too general. If $X$ is uniform on $\sqrt{n}B_2^n$, and $f(x) = \frac{1}{\sqrt{n}}\|x\|^2$, an easy calculation shows,

$$\text{Var}(f(X)) \approx \frac{1}{n}.$$ 

More generally, if $X$ is uniform on an isotropic $L_p$ ball, and $f(x) = \frac{1}{\sqrt{n}}\|x\|^p_p$,

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There is also a connection between anti-concentration and Fourier transforms that goes back to Esseen:

**Lemma (Esseen’s inequality)**

*Let* $X$ *be a random variable with characteristic function* $\varphi$, *then for any* $\varepsilon > 0$ *and* $t \in \mathbb{R}$,*

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P(|X - t| < \varepsilon) \leq \varepsilon \int_{-2\pi/\varepsilon}^{2\pi/\varepsilon} |\varphi(\lambda)| d\lambda.
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In particular, if $|\varphi(\lambda)| \lesssim \frac{1}{|\lambda|^\alpha}$, $P(|X - t| < \varepsilon) \leq \varepsilon^\alpha$. 
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In particular, if $|\varphi(\lambda)| \lesssim \frac{1}{|\lambda|^\alpha}$, $\mathbb{P}(|X - t| < \varepsilon) \leq \varepsilon^\alpha$. 
Recall the classical Van der Corput lemma from the 30’s. If \( h : \mathbb{R} \rightarrow \mathbb{R} \), is such that \( |h^{(k)}| \geq 1 \). Then,

\[
\int_{-1}^{1} e^{i\lambda h(x)} \, dx \lesssim \frac{1}{|\lambda|^{\frac{1}{k}}}.
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In particular, if \( X \) is uniform on \([-1, 1]\) and \( f(x) = x^d \),

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\mathbb{P}(|f(X) - t| < \varepsilon) \lesssim \varepsilon^\frac{1}{d}.
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What happens in higher dimensions?
Recall the classical Van der Corput lemma from the 30’s. If $h : \mathbb{R} \to \mathbb{R}$, is such that $|h^{(k)}| \geq 1$. Then,

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What happens in higher dimensions?
Much work has been done on high-dimensional analogues of the Van der Corput lemma. Carbery, Christ and Wright showed,

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\int_{[-1,1]^n} e^{i\lambda f(x)} \, dx \lesssim \frac{\text{poly}(n)}{|\lambda|^{1/d}},
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if \( f \) is a homogeneous degree \( d \) polynomial and for some \( I \), \(|v_I| \geq 1\).

They also asked the question: can the dependence on \( n \) be removed from the right hand side?
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They also asked the question: can the dependence on $n$ be removed from the right hand side?
Our results

**Theorem**

Let $X \sim \mu \otimes^n$ be a product measure and let $f(x) = \sum_i v_i x^I$ with $\sum v_i^2 = 1$. Then,  

$$\text{Var}(f(X)) \geq C_{\mu,d}.$$  

Moreover, the constant can be taken to be uniform over all isotropic log-concave product measures.

**Corollary**

Let $f(x) = \sum_i v_i x^I$ with $|v_i| \geq 1$ for some $I$. Then,  

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Let $X \sim \mu \otimes^n$ be a product measure and let $f(x) = \sum_{l} v_l x^l$ with $\sum v_l^2 = 1$. Then,

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for $|\lambda| > 1$, with $\lesssim$ denoting that the left-hand side is bounded by the right-hand side up to a constant factor independent of $\lambda$. The constant of proportionality is uniform over all isotropic log-concave product measures.
Let $X \sim \mu$ be random variable with infinite support. Apply the Gram-Schmidt algorithm to $\{1, x, x^2, \ldots \}$ in $L^2(\mu)$ and consider the resulting orthogonal polynomials $\{p_k\}_{k=0}^{\infty}$.

**Lemma**

Let $f(x) = x^d$. Then,

1. $\langle f, p_k \rangle_{L^2(\mu)} = 0$ for $k > d$.
2. $\langle f, p_d \rangle_{L^2(\mu)} = \tilde{c}_{\mu,d} \neq 0$.

**Proof.**

1. $p_k$ is orthogonal to degree $d$ polynomials.
2. $f \notin \text{span} \{1, x, x^2, \ldots, x^{d-1}\}$. 
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□
Observe $L^2(\mu^\otimes n) = L^2(\mu)^\otimes n$. So, an orthonormal basis for $L^2(\mu^\otimes n)$ is given by $\{p_I\}$, where for $I = (I_1, I_2, \ldots, I_n)$,

$$p_I(x) = \prod_{i=1}^{n} p_{I_i}(x_i).$$

**Lemma**

Let $f(x) = \sum_I v_I x^I$, be of degree $d$. Then, for $|J| = d$,

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Proof.

\[
\langle f, p_J \rangle_{L^2(\mu^\otimes n)} = \sum v_I \langle x^I, p_J \rangle_{L^2(\mu^\otimes n)} = v_J \langle x^J, p_J \rangle_{L^2(\mu^\otimes n)} \\
= v_J \prod_{i=1}^d \langle x^J_i, p_J_i \rangle_{L^2(\mu)} = v_J \prod_{i=1}^d \tilde{c}_{\mu,J_i}.
\]

An $L^2$ decomposition gives

\[
\text{Var}(f(X)) = \sum_{l \neq 0} \langle f, p_l \rangle_{L^2(\mu^\otimes n)}^2.
\]

and we are ready to prove the theorem.
Dimension-free variance bounds

Proof.

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\langle f, p_J \rangle_{L^2(\mu^\otimes n)} = \sum v_I \langle x^I, p_J \rangle_{L^2(\mu^\otimes n)} = v_J \langle x^J, p_J \rangle_{L^2(\mu^\otimes n)}
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Proof of Theorem.

\[
\text{Var}(X) = \left\langle f, f \right\rangle_{L^2(\mu \otimes n)} - \left\langle 1, f \right\rangle_{L^2(\mu \otimes n)}^2 = \sum_{l \neq 0} \left\langle f, p_l \right\rangle_{L^2(\mu \otimes n)}^2 \\
\geq \sum_{|l| = d} \left\langle f, p_l \right\rangle_{L^2(\mu \otimes n)}^2 \geq c_{\mu, d}^2 \sum_{|l| = d} v_l^2 \\
= c_{\mu, d}^2.
\]

When \( \mu \) is log-concave isotropic, by a comparison to an interval, we get \( c_{\mu, d} = c^d \).
We can now combine our result with the Carbery-Wright inequality.

**Corollary**

Let $X$ be a log-concave with a product law and let $f(x) = \sum v_l x^l$, be of degree $d$. Then, for any $\varepsilon > 0$ and $t \in \mathbb{R}$,

$$\mathbb{P}\left( |f(X) - t| \leq \varepsilon \right) \lesssim \varepsilon^{\frac{1}{d}}.$$

**Proof.**

$$\mathbb{P}\left( |f(X) - t| \leq \varepsilon \right) \lesssim \left( \frac{\varepsilon}{\sqrt{\mathbb{E}[f(X)^2]}} \right)^{\frac{1}{d}} \leq \left( \frac{\varepsilon}{c^d} \right)^{\frac{1}{d}}.$$
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Let \( f(x) = \sum_{l} v_l x^l \) with \( |v_l| \geq 1 \) for some \( l = (l_1, \ldots, l_n) \). We wish to bound,

\[
J(\lambda) := \int_{[-1,1]^n} e^{i\lambda f(x)} \, dx.
\]

Define,

\[
A := \left\{ x \in [-1,1]^n \mid \left| \frac{d}{dx_{l_n}} f(x) \right| \geq \varepsilon \right\}.
\]

So,

\[
J(\lambda) \leq \left| \int_{A} e^{i\lambda f(x)} \, dx \right| + \left| \int_{\bar{A}} e^{i\lambda f(x)} \, dx \right|.
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Let $f(x) = \sum_{l} v_l x^l$ with $|v_l| \geq 1$ for some $l = (l_1, \ldots, l_n)$. We wish to bound,

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We first bound $\bar{A}$.

The main observation is that $\frac{d}{dx_n} f$ is a polynomial of degree $d - l_n$ with sum of coefficients at least 1,

$$\left| \int_{\bar{A}} e^{i\lambda f(x)} dx \right| \leq \int_{\bar{A}} 1 dx = \mathbb{P}\left( \left| \frac{d}{dx_n} f(X) \right| \leq \varepsilon \right) \lesssim \varepsilon^{\frac{1}{d-l_n}}.$$
We also bound

\[ \left| \int_{\tilde{A}} e^{i\lambda f(x)} \, dx \right| \lesssim \frac{1}{(|\lambda| \varepsilon)^{\frac{1}{10n}}} \cdot \]

High-level idea:

- Decompose \( x = (\tilde{x}, x_n) \) and \( f_{\tilde{x}}(x_n) = f(x) \).
- On \( A \), for every \( \tilde{x} \), \( |f_{\tilde{x}}^{(I_n)}| \geq \varepsilon \).
- Use one-dimensional results for \( f_{\tilde{x}} \).
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Multivariate Van der Corput

\[ J(\lambda) \leq \left| \int_A e^{i\lambda(x)} \, dx \right| + \left| \int_{\bar{A}} e^{i\lambda(x)} \, dx \right| \]

\[ \leq \frac{1}{\left( |\lambda| \varepsilon \right) \frac{1}{ln}} + \varepsilon \frac{1}{d-ln} \cdot \]

Optimize over \( \varepsilon \) to get,

\[ J(\lambda) \lesssim \frac{1}{|\lambda| \frac{1}{d}} \cdot \]

**Question**

Is the condition \( |v_I| \geq 1 \) necessary?
Multivariate Van der Corput

\[ J(\lambda) \leq \left| \int_A e^{if\lambda(x)} \, dx \right| + \left| \int_{\bar{A}} e^{if\lambda(x)} \, dx \right| \leq \frac{1}{(|\lambda|\varepsilon)^{\frac{1}{ln}}} + \varepsilon^{\frac{1}{d-ln}}. \]

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**Question**

Is the condition \( |v_I| \geq 1 \) necessary?
Beyond products

Recall that if $X \sim \text{Uniform}(B^n_p)$ and $f(x) = \frac{1}{\sqrt{n}}\|x\|_p^p$,

$$\text{Var}(f(X)) = o(1).$$

However, in these cases we have,

$$\mathbb{E} [f(X)^2] = \omega(1).$$

Can we get dimension-free estimates on $\mathbb{E} \left[ (X \otimes d) (X \otimes d)^T \right]$, instead of $\text{Cov} (X \otimes d)$?

Is $\mathbb{E} [f(X)^2]$ large, when $\sum \nu_i^2 = 1$?
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Is $\mathbb{E} \left[ f(X)^2 \right]$ large, when $\sum \nu_i^2 = 1$?
Let $Y \sim \frac{1}{Z} e^{-\|x\|_p^p} dx$ and $U \sim \text{Uniform}([0, 1])$. Then

$$X = n^{\frac{1}{p}} U \frac{Z}{\|Z\|_p}.$$

Since $Z$ is a product measure, for any homogeneous function,

$$\mathbb{E} \left[ f(X)^2 \right] \approx n^{\frac{2d}{p}} \mathbb{E} \left[ \frac{f(Z)^2}{\|Z\|_p^{2d}} \right] \geq 1.$$
Euclidean balls

1. If \(X\) is uniform on the isotropic Euclidean ball, we identify all eigenvalues of \(\text{Cov}(X \otimes^d)\).

2. Eigenvectors are given by \(\|x\|^{2k}_2 H_{d-2k}\), where \(H_{d-2k}\) are degree \(d-2k\) spherical harmonics.

3. If \(f(x) = \|x\|^2_2\), \(\text{Var}(f(X)) \approx \frac{1}{n}\).

4. If \(f\) is orthogonal to \(\|x\|^2_2\), \(\text{Var}(f(X)) = \Omega(1)\).
Thank You