EXERCISE 5 IN COMMUTATIVE ALGEBRA AND ALGEBRAIC GEOMETRY II

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(1) (P) Show that for an object I in an abelian category C, the following are equivalent.
  (a) I is injective
  (b) Any monomorphism $I \hookrightarrow N$ has a section
  (c) Any exact sequence $0 \to I \to N \to K \to 0$ splits.
(2) Let X be an affine variety. Define a pre-sheaf $\mathcal{O}'_X(U) := \{\frac{f}{g} \mid f, g \in \text{Pol}(X), g \text{ doesn't vanish on } U\}$. Show that $sh(\mathcal{O}'_X) = \mathcal{O}_X$.
(3) The functor of global sections is left-exact, but not exact
(4) $0 \to \mathcal{O}_{\mathbb{P}^1}(\infty) \to \mathcal{O}_{\mathbb{P}^1} \to k_\infty \to 0$, where k_∞ is the 1-dimensional skyscraper at the point $\infty \in \mathbb{P}^1$.
(5) (a) $D_1 \equiv D_2 \Rightarrow \mathcal{O}(D_1) \cong \mathcal{O}(D_2)$
  (b) $\mathcal{O}(D_1) \otimes \mathcal{O}(D_2) \cong \mathcal{O}(D_1 + D_2)$
  (c) $\mathcal{O}(D)\check{} \cong \mathcal{O}(-D)$
(6) The group $Pic(C)$ is isomorphic to the group of invertible sheaves, with tensor product as multiplication.
(7) Let $X = SpecA$ be an algebraic variety, M be an A-module and $\tilde{M}$ be the corresponding quasi-coherent sheaf. Show that $\tilde{M}(X) = M$ and $\tilde{M}_p = M_{m_p}$ where $p \in X$ is any point and $m_p \subset A$ the corresponding maximal ideal.
(8) Give an example of a non-zero module over $k[t_0, t_1]$ such that the corresponding sheaf on $\mathbb{P}^1$ is zero.

URL: http://www.wisdom.weizmann.ac.il/~dimagur/AlgGeo.html

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