Fourier coefficients of automorphic forms & applications to minimal and next-to-minimal forms.

Dmitry Gourevitch
Weizmann Institute of Science, Israel
http://www.wisdom.weizmann.ac.il/~dimagur

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j.w. H. P. A. Gustafsson, A. Kleinschmidt, D. Persson, and S. Sahi

Following Piatetski-Shapiro–Shalika, Ginzburg–Rallis–Soudry, Moeglin-Waldspurger, Jiang–Liu–Savin, Gomez, Ahlen, Hundley–Sayag, Green-Miller-Vanhove, Kazhdan–Polishchuk, Bossard–Pioline

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Definitions

- $\mathbb{K}$: number field, $\mathbb{A} := \mathbb{A}_{\mathbb{K}}$, $\mathbf{G}$: reductive group over $\mathbb{K}$, $\Gamma := \mathbf{G}(\mathbb{K})$, $G := \mathbf{G}(\mathbb{A})$, $\mathfrak{g} := Lie(\Gamma)$.
- Fix a semisimple $H \in \mathfrak{g}$, and let $\mathfrak{g}_i := \mathfrak{g}_i^H$ denote the eigenspaces of $ad(H)$. Assume that all the eigenvalues i lie in $\mathbb{Q}$.
- Let $f \in \mathfrak{g}_{-2}$. Call $(H, f) \in \mathfrak{g} \times \mathfrak{g}$ a *Whittaker pair*.
- Define $\mathfrak{n} := \mathfrak{n}_{H,f} := (\mathfrak{g}_1 \cap \mathfrak{g}^f) \oplus \bigoplus_{i>1} \mathfrak{g}_i$, $N := \operatorname{Exp}(\mathfrak{n})(\mathbb{A})$.
- Fix a non-trivial unitary additive character $\psi : \mathbb{K}\backslash\mathbb{A} \to \mathbb{C}$ and define $\chi_f : N \to \mathbb{C}$ by $\chi_f(\operatorname{Exp} X) := \psi(\langle f, X\rangle)$.
- $C^\infty(\Gamma\backslash G) :=$ functions on $\Gamma\backslash G$ smooth on G_∞ and finite under K_{fin}.
- Let $[N] := (\Gamma \cap N)\backslash N$. For $\eta \in C^\infty(\Gamma\backslash G)$, define *Fourier coefficient*

$$\mathcal{F}_{H,f}[\eta](g) := \int_{[N]} \eta(ng)\chi_f(n)^{-1} dn.$$

Two central cases of Fourier coefficients

$[H,f]=-2f$, $\mathfrak{n}=(\mathfrak{g}_1\cap\mathfrak{g}^f)\oplus\bigoplus_{i>1}\mathfrak{g}_i$, $N=\mathrm{Exp}(\mathfrak{n})(\mathbb{A})$, $\eta\in C^\infty(\Gamma\backslash G)$,

$$\mathcal{F}_{H,f}[\eta](g):=\int_{[N]}\eta(ng)\chi_f(n)^{-1}dn.$$

- Neutral Fourier coefficient, coming from $\mathfrak{sl}_2$-triple (e,H,f), *e.g.*:

$$H=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}\ f=\begin{pmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&1&0\end{pmatrix}\ \mathfrak{n}=\begin{pmatrix}0&\underline{*}&0&*\\0&0&0&0\\0&*&0&\underline{*}\\0&0&0&0\end{pmatrix}$$

- Whittaker coefficient $\mathcal{W}_f$, with N maximal unipotent, *e.g.*:

$$H=\begin{pmatrix}3&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-3\end{pmatrix}\ f=\begin{pmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&1&0\end{pmatrix}\ \mathfrak{n}=\begin{pmatrix}0&\underline{*}&*&*\\0&0&*&*\\0&0&0&\underline{*}\\0&0&0&0\end{pmatrix}$$

Coefficients that are both neutral and Whittaker are Eulerian by local uniqueness of Whittaker models.

Examples of Fourier coefficients

$[H, f] = -2f$, $\mathfrak{n} = (\mathfrak{g}_1 \cap \mathfrak{g}^f) \oplus \bigoplus_{i>1} \mathfrak{g}_i$, $N = \operatorname{Exp}(\mathfrak{n})(\mathbb{A})$, $\eta \in C^\infty(\Gamma \backslash G)$,

$$\mathcal{F}_{H,f}[\eta](g) := \int_{[N]} \eta(ng)\chi_f(n)^{-1} dn.$$

Comparison for $G = \mathrm{GL}_3(\mathbb{K})$:

- Neutral Fourier coefficient:

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \mathfrak{n} = \begin{pmatrix} 0 & 0 & \underline{*} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- Whittaker coefficient:

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \mathfrak{n} = \begin{pmatrix} 0 & * & \underline{*} \\ 0 & 0 & 0 \\ 0 & * & 0 \end{pmatrix}$$

Root exchange

- $\mathfrak{u} := \mathfrak{g}_1/(\mathfrak{g}_1 \cap \mathfrak{g}^f)$. $\omega_f(X, Y) := \langle f, [X, Y] \rangle$- symplectic form.
- $\forall$ isotropic subspace $\mathfrak{i} \subset \mathfrak{u}$, let $I := \operatorname{Exp}(\mathfrak{i})(\mathbb{A})$ and

$$\mathcal{F}^I_{H,f}[\eta](g) := \int_{[I]} \mathcal{F}_{H,f}[\eta](ug)\, du$$

Lemma

(i) $\mathcal{F}_{H,f}[\eta](g) = \sum_{\gamma \in (U/I^\perp)(\mathbb{K})} \mathcal{F}^I_{H,f}[\eta](\gamma g)$

(ii) *For any isotropic subspace* $\mathfrak{j} \subset \mathfrak{u}$ *with* $\dim \mathfrak{j} = \dim \mathfrak{i}$ *and* $\mathfrak{j} \cap \mathfrak{i}^\perp = \{0\}$,

$$\mathcal{F}^J_{H,f}[\eta](g) = \int_{J(\mathbb{A})} \mathcal{F}^I_{H,f}[\eta](ug)\, du$$

For $H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$: $\begin{pmatrix} 0 & \mathfrak{i} & \underline{\mathfrak{n}} \\ 0 & 0 & \mathfrak{j} \\ 0 & 0 & 0 \end{pmatrix}$

Relating different coefficients

- $\mathrm{WO}(\eta) := \{\mathcal{O} \in \mathcal{N}(\mathfrak{g}) \mid \forall$ neutral (h, f) with $f \in \mathcal{O}$, $\mathcal{F}_{h,f}(\eta) \not\equiv 0\}$.
- Say $(H, f) \succ (S, f)$ if $[H, S] = 0$ and $\mathfrak{g}^f \cap \mathfrak{g}^H_{\geq 1} \subseteq \mathfrak{g}^{S-H}_{\geq 0}$.
- f is $\mathbb{K}$-distinguished if $\forall$ Levi $\mathfrak{l} \ni f$ defined over $\mathbb{K}$, $\mathfrak{l} = \mathfrak{g}$.
- (S, f) is called Levi-distinguished if $\exists$ parabolic $\mathfrak{p} = \mathfrak{l}\mathfrak{u}$ s.t. f is $\mathbb{K}$-distinguished in $\mathfrak{l}$, and $\mathfrak{n}_{S,f} = \mathfrak{l}_{S,f} \oplus \mathfrak{u}$.
- Whittaker coefficients are Levi-distinguished.
- For Whittaker pairs with the same f and commuting H-s, neutral $\succ$ any $\succ$ Levi-distinguished.

Theorem

Let $(H, f) \succ (S, f)$. Then

(i) *$\mathcal{F}_{H,f}[\eta]$ linearly determines $\mathcal{F}_{S,f}[\eta]$.*

(ii) *If $\Gamma f \in \mathrm{WO}^{\max}(\eta)$ and $\mathfrak{g}^H_1 = \mathfrak{g}^S_1 = 0$ let $\mathfrak{v} := \mathfrak{g}^H_{>1} \cap \mathfrak{g}^S_{<1}$. Then*

$$\mathcal{F}_{H,f}[\eta](g) = \int_{V(\mathbb{A})} \mathcal{F}_{S,f}[\eta](vg)\, dv$$

Theorem

Let $(H,f) \succ (S,f)$. *Then*

(i) $\mathcal{F}_{H,f}[\eta]$ *linearly determines* $\mathcal{F}_{S,f}[\eta]$.

(ii) *If* $\Gamma f \in \mathrm{WO}^{\max}(\eta)$ *and* $\mathfrak{g}_1^H = \mathfrak{g}_1^S = 0$ *let* $\mathfrak{v} := \mathfrak{g}^H_{>1} \cap \mathfrak{g}^S_{<1}$. *Then*

$$\mathcal{F}_{H,f}[\eta](g) = \int_{V(\mathbb{A})} \mathcal{F}_{S,f}[\eta](vg)\, dv$$

Corollary

(i) *If* η *is cuspidal then any* $\mathcal{O} \in \mathrm{WO}^{\max}(\eta)$ *is* $\mathbb{K}$*-distinguished.*
In particular, $\mathcal{O}$ *is totally even for* $G = \mathrm{Sp}_{2n}$, *totally odd for* $G = SO(V)$, *not minimal for* $\mathrm{rk} G > 1$,
and not next-to-minimal for $\mathrm{rk} G > 2$, $G \neq F_4$.

(ii) *If* $f \notin \mathrm{WO}(\eta)$ *then* $\mathcal{F}_{H,f}(\eta) = 0$ *for any* H.

(iii) *Let* G *be simply-laced,* H *define a maximal parabolic,* $f \in \mathrm{WO}^{\max}(\eta)$:
f *is minimal* $\Rightarrow \mathcal{F}_{H,f}(\eta) = \mathcal{W}_f(\eta)$
f *is next-to-minimal* $\Rightarrow \mathcal{F}_{H,f}(\eta) = \int_{V(\mathbb{A})} \mathcal{W}_f(\eta)$.
The RHS is frequently Eulerian.

Example

$G := \mathrm{GL}(4, \mathbb{A})$, $f := E_{21} + E_{43}$, $H := \mathrm{diag}(3, 1, -1, -3)$,
$h = \mathrm{diag}(1, -1, 1, -1)$, $Z = H - h = \mathrm{diag}(2, 2, -2, -2)$, $H_t := h + tZ$.
Then $\mathfrak{n}_0 \subset \mathfrak{n}_{1/4} \oplus \mathfrak{i} \sim \mathfrak{n}_{1/4} \oplus \mathfrak{j} \subset \mathfrak{n}_{3/4} = \mathfrak{n}_1$:

$$\begin{pmatrix} 0 & - & 0 & - \\ 0 & 0 & 0 & 0 \\ 0 & - & 0 & - \\ 0 & 0 & 0 & 0 \end{pmatrix} \subset \begin{pmatrix} 0 & - & a & - \\ 0 & 0 & 0 & a \\ 0 & * & 0 & - \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & - & * & - \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & - \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\subset \begin{pmatrix} 0 & - & - & - \\ 0 & 0 & * & - \\ 0 & 0 & 0 & - \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & - & - & - \\ 0 & 0 & - & - \\ 0 & 0 & 0 & - \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Both $*$ and $-$ denote arbitrary elements. $-$ denotes the entries in $\mathfrak{g}^{H_t}_{>0}$ and $*$ those in $\mathfrak{g}^{H_t}_1$. a denotes equal elements in $\mathfrak{g}^{H_t}_1 \cap \mathfrak{g}^f$.

Expressing forms through their Whittaker coefficients

Theorem

Any $\mathcal{F}_{H,f}$ is linearly determined by all Levi-distinguished Fourier coefficients $\mathcal{F}_{S,F}$ with $\Gamma F \geq \Gamma f$.

Corollary

(i) *Any $\eta \in C^\infty(\Gamma \backslash G)$ is linearly determined by all its Levi-distinguished Fourier coefficients.*

(ii) *If all $\mathcal{O} \in \mathrm{WO}(\eta)$ admit Whittaker coefficients then η is linearly determined by its Whittaker coefficients.*

(iii) *If G is split and simply-laced, and η is minimal or next-to-minimal then all Fourier coefficients of η are linearly determined by Whittaker coefficients of η.*

Parabolic minimal Fourier coeff. of next-to-minimal forms

- $\mathfrak{g}$ split simply laced, $\mathfrak{h} \subset \mathfrak{g}$ Cartan, $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{u}$ Borel
- α simple root. $\mathfrak{q}_\alpha = \mathfrak{l}_\alpha \oplus \mathfrak{n}_\alpha = \mathfrak{g}^{S_\alpha}_{\geq 0}$ max. parabolic.
- $I^{(\perp\alpha)} = \{\beta_1, \ldots, \beta_k\}$ Bourbaki enumeration of the simple roots orthogonal to α.
- $\forall i \; G \supset L_i :=$ Levi given by roots $\beta_1, \ldots, \beta_i$
- $L_i \supset S_i :=$ stabilizer of the root space $\mathfrak{g}_{-\beta_i}$, $\Gamma_i := (L_i \cap \Gamma)/(S_i \cap \Gamma)$.
- For $f \in \mathfrak{g}^\times_{-\alpha}$ and next-to-minimal $\eta_{\mathrm{ntm}} \in C^\infty(\Gamma \backslash G)$ let

$$A_i^f[\eta_{\mathrm{ntm}}](g) := \sum_{\gamma \in \Gamma_{i-1}} \sum_{\varphi \in \mathfrak{g}^\times_{-\beta_i}} \mathcal{W}_{\varphi+f}[\eta_{\mathrm{ntm}}](\gamma g)$$

Theorem

$$\mathcal{F}_{S_\alpha, f}[\eta_{\mathrm{ntm}}] = \mathcal{W}_f[\eta_{\mathrm{ntm}}] + \sum_{i=1}^{k} A_i^f[\eta_{\mathrm{ntm}}]$$

Explanation for GL_n (PS-Shalika, Ahlen–Gustafsson-Liu-Kleinschmidt-Persson)

Let $\eta \in C^\infty(\Gamma\backslash \mathrm{GL}_n(\mathbb{A}))$. Restrict to the last column and decompose to Fourier series. All non-trivial characters are conjugate by $\mathrm{GL}_{n-1}(\mathbb{K})$.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & \underline{*} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Explanation for GL_n

Let $\eta \in C^\infty(\Gamma\backslash \mathrm{GL}_n(\mathbb{A}))$. Restrict to the last column and decompose to Fourier series. All non-trivial characters are conjugate by $\mathrm{GL}_{n-1}(\mathbb{K})$. Conjugate, restrict to the next column and continue

$$\begin{pmatrix} 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & \underline{*} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & \underline{*} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & \underline{*} & * \\ 0 & 0 & 0 & 0 & \underline{*} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \cdots$$

- For η_{min}, only one column can have a non-trivial character.
- For η_{ntm}, $\mathcal{F}_{S_\alpha,f}$ is a minimal automorphic function on L_α.

$$\begin{pmatrix} 0 & 0 & * & * & * \\ 0 & 0 & \underline{*} & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & * & * & * & * \\ 0 & 0 & \underline{*} & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & * & * & * & * \\ 0 & 0 & \underline{*} & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & \underline{*} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

For η_{ntm}, $\mathcal{F}_{S_\alpha,f}$ is a minimal automorphic function on L_α.

$$\begin{pmatrix} 0 & 0 & * & * & * \\ 0 & 0 & \underline{*} & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & * & * & * & * \\ 0 & 0 & \underline{*} & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & * & * & * & * \\ 0 & 0 & \underline{*} & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & \underline{*} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{F}_{S_\alpha,f}[\eta_{\mathrm{ntm}}] = \mathcal{W}_f[\eta_{\mathrm{ntm}}] + \sum_{i=1}^{k} A_i^f[\eta_{\mathrm{ntm}}]$$

where $f \in \mathfrak{g}^\times_{-\alpha}$ and

$$A_i^f[\eta](g) := \sum_{\gamma\in\Gamma_{i-1}} \sum_{\varphi\in\mathfrak{g}^\times_{-\beta_i}} \mathcal{W}_{\varphi+f}[\eta](\gamma g)$$