Finite multiplicities beyond spherical pairs

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Basic Functions, Orbital Integrals, and Beyond Endoscopy
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arXiv:2109.00204

BIRS, November 2021
\( \mathbf{G} \): reductive group over \( \mathbb{R} \), \( \mathbf{X} := \) algebraic \( \mathbf{G} \)-manifold, \( \mathfrak{g} := \text{Lie}(\mathbf{G}) \), \( \mathcal{N}(\mathfrak{g}^*) \):=nilpotent cone, \( G := \mathbf{G}(\mathbb{R}) \), \( X := \mathbf{X}(\mathbb{R}) \),
\(G\): reductive group over \(\mathbb{R}\), \(X:=\) algebraic \(G\)-manifold, \(\mathfrak{g}:=\text{Lie}(G)\), \(\mathcal{N}(\mathfrak{g}^*):=\) nilpotent cone, \(G:=G(\mathbb{R})\), \(X:=X(\mathbb{R})\),
\(S(X):=\) smooth functions on \(X\), flat at infinity (Schwartz).
G: reductive group over \( \mathbb{R} \), \( X := \text{algebraic } G \)-manifold, \( g := \text{Lie}(G) \), \( \mathcal{N}(g^*) := \text{nilpotent cone} \), \( G := G(\mathbb{R}) \), \( X := X(\mathbb{R}) \).

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- Major Goal: study \( L^2(X), C^\infty(X), S(X) \) as rep-s of \( G \).

Studied by Bernstein, Delorme, van den Ban, Schlichtkrull, Kroetz, Kobayashi, Oshima, Knop, Beuzart-Plessis, Kuit, Wan, ...

Theorem (Kobayashi-Oshima, 2013)

Let \( X = G/H \). Then

1. \( X \) is spherical \( \iff \) \( S(X) \) has bounded multiplicities.
2. \( X \) is real-spherical \( \iff \) \( S(X) \) has finite multiplicities.

\[ m_\sigma(S(X)) := \dim \text{Hom}(S(X), \sigma), \]

\[ m_\sigma(S(G/H)) := \dim(\sigma^- \mathcal{H}) \]

Theorem (Casselman, 1978)

\[ 0 < m_\sigma(S(G/U)) < \infty \quad \forall \sigma \in \text{Irr}(G), \]

where \( U = \text{maximal unipotent} \).
- $G$: reductive group over $\mathbb{R}$, $X :=$ algebraic $G$-manifold, $\mathfrak{g} := \text{Lie}(G)$, $\mathcal{N}(\mathfrak{g}^*) :=$ nilpotent cone, $G := G(\mathbb{R})$, $X := X(\mathbb{R})$,
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∀x ∈ X, have action map G → X, thus g → T_xX, and T^*_xX → g^*.
This gives the moment map μ : T^*X → g^*.
For X = G/H : T^*X ≅ G ×_H h^⊥ and μ(g, α) = g · α
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For \( X = G/H : T^*X \cong G \times_H \mathfrak{h}^\perp \) and \( \mu(g, \alpha) = g \cdot \alpha \).

**Definition**

- For a nilpotent orbit \( O \subset \mathcal{N}(g^*) \), say \( X \) is \( O \)-spherical if
  \[
  \dim \mu^{-1}(O) \leq \dim X + \dim O / 2
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- For a \( G \)-invariant subset \( \Xi \subset \mathcal{N}(g^*) \), say \( X \) is \( \Xi \)-spherical if \( X \) is \( O \)-spherical \( \forall O \subset \Xi \).

For \( X = G/H \), \( X \) is \( O \)-spherical \( \iff \) \( \dim O \cap \mathfrak{h}^\perp \leq \dim O / 2 \).
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**Definition**

*For a nilpotent orbit \( \mathcal{O} \subset \mathcal{N}(\mathfrak{g}^*) \), say \( X \) is \( \mathcal{O} \)-spherical if*

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For \( X = G/H \), \( X \) is \( \mathcal{O} \)-spherical \( \iff \) \( \dim \mathcal{O} \cap \mathfrak{h}^\perp \leq \dim \mathcal{O}/2 \).

For parabolic \( P \subset G \), \( \mathcal{O}_P \) := the unique orbit s.t. \( p^\perp \cap \mathcal{O}_P \) is dense in \( p^\perp \).

**Theorem 1 (Aizenbud - G. 2021)**

\( X \) is \( \overline{\mathcal{O}_P} \)-spherical \( \iff \) \( P \) has finitely many orbits on \( X \).
∀x ∈ X, have action map G → X, thus g → T_xX, and T_xX → g^*.
This gives the moment map μ : T^*X → g^*.
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Definition

- For a nilpotent orbit O ⊂ N(g^*), say X is O-spherical if
  \[\dim \mu^{-1}(O) \leq \dim X + \dim O/2\]

- For a G-invariant subset Ξ ⊂ N(g^*), say X is Ξ-spherical if X is O-spherical ∀O ⊂ Ξ.

For X = G/H, X is O-spherical ⇐⇒ \(\dim O \cap h⊥ \leq \dim O/2\).
For parabolic P ⊂ G, OP := the unique orbit s.t. \(p⊥ \cap OP\) is dense in \(p⊥\).

Theorem 1 (Aizenbud - G. 2021)

X is OP-spherical ⇐⇒ P has finitely many orbits on X.

Corollary (following Wen-Wei Li)

- X is N(g^*)-spherical ⇐⇒ X is spherical
- X is \(\{0\}\)-spherical ⇐⇒ G has finitely many orbits on X.
**Associated variety of the annihilator & the main theorem**

- $\mathcal{U}_n(g)$ - PBW filtration on universal enveloping algebra.
Associated variety of the annihilator & the main theorem

- $\mathcal{U}_n(g)$ - PBW filtration on universal enveloping algebra.
- $\text{gr}\mathcal{U}(g) \cong S(g) \cong \text{Pol}(g^*)$. 

Theorem 2 (Aizenbud - G. 2021)

Let $\Xi \subset \mathbb{N}(g^*)$ closed $G$-invariant. Let $X$ be $\Xi$-spherical $G$-manifold, and let $\sigma \in \mathcal{M}_\Xi(G)$. Then $\dim \text{Hom}(S(X), \sigma) < \infty$.
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**Theorem 2 (Aizenbud - G. 2021)**

*Let \( \mathfrak{E} \subset \mathcal{N}(g^*) \) closed \( G \)-invariant. Let \( X \) be \( \mathfrak{E} \)-spherical \( G \)-manifold, and let \( \sigma \in \mathcal{M}_\mathfrak{E}(G) \). Then \( \dim \text{Hom}(S(X), \sigma) < \infty \)*
Corollary

Let $H \subset G$ be reductive subgroup. Let $P \subset G$ and $Q \subset H$ be parabolic subgroups with $|P \backslash G / Q| < \infty$. Then $\forall \pi \in M_{Op}(G)$ and $\tau \in M_{OQ}(H)$,

$$\dim \text{Hom}_H(\pi|_H, \tau) < \infty$$
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Corollary

(i) Let $P \subset G$ be a parabolic subgroup such that $G/P$ is a spherical $H$-variety. Then $\forall \pi \in M_{Op}(G)$, $\pi|_H$ has finite multiplicities.

(ii) Let $Q \subset H$ be a parabolic subgroup that is spherical as a subgroup of $G$. Then for any $\tau \in M_{OQ}(H)$, $\text{ind}_H^G \tau$ has finite multiplicities.
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For simple $G$ and symmetric $H \subset G$, all $P \subset G$ satisfying (i), and all $Q \subset H$ satisfying (ii) are classified by He, Nishiyama, Ochiai, Oshima. For classical $G$, all $H$: Avdeev-Petukhov. They also have a strategy $\forall G$. 
Corollary

Let $H$ be a reductive group, and $P, Q \subset H$ be parabolic subgroups s.t. $H/P \times H/Q$ is a spherical $H$-variety, under the diagonal action. Then $\forall \pi \in \mathcal{M}_{O_P}(H)$, and $\tau \in \mathcal{M}_{O_Q}(H)$, $\pi \otimes \tau$ has finite multiplicities.

All such triples $(H, P, Q)$ were classified by Stembridge.
Example: $H = GL_n$, $\tau \in \mathcal{M}_{O_{\text{min}}}(H)$, or classical $H$ and $\pi, \tau \in \mathcal{M}_{O_{2n}}(H)$.
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- Our results also extend to certain representations of non-reductive \( H \).
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Example: $H = GL_n$, $\tau \in \mathcal{M}_{\mathcal{O}_{\text{min}}}(H)$, or classical $H$ and $\pi, \tau \in \mathcal{M}_{\mathcal{O}_{2^n}}(H)$.

- Our results also extend to certain representations of non-reductive $H$.

Example (Generalized Shalika model)

Let $G = GL_{2n}$, $R = LU \subset G$ with $L = GL_n \times GL_n$ and $U = \text{Mat}_{n \times n}$, $M = \Delta GL_n \subset L$, $H := MU$.

Let $m^* \supset \mathcal{O}_{\text{min}} := \text{minimal nilpotent orbit}$, and $\pi \in \mathcal{M}_{\mathcal{O}_{\text{min}}}(M)$.

Let $\psi$ be a unitary character of $H$.

Then $\text{ind}_H^G(\pi \otimes \psi)$ has finite multiplicities.

Similar case: $G = O_{4n}$, $L = GL_{2n}$, $M = \text{Sp}_{2n}$, $\mathcal{O}_{\text{ntm}} \subset m^*$. 
Some necessary conditions for finite multiplicities

**Theorem (Tauchi)**

Let $P \subset G$ be a parabolic subgroup. If all degenerate principal series representations of the form $\text{Ind}_P^G \rho$, with $\dim \rho < \infty$, have finite $H$-multiplicities, then $H$ has finitely many orientable orbits on $G/P$. 

**Corollary**

Let $P \subset G$ be a parabolic subgroup defined over $\mathbb{R}$. Suppose that for all but finitely many orbits of $H$ on $G/P$, the set of real points is non-empty and orientable. Then the following are equivalent.

(i) $H$ is $O_P$-spherical.

(ii) Every $\pi \in M_{O_P}(G)$ has finite multiplicities in $S(G/H)$.

(iii) $H$ has finitely many orbits on $G/P$.

(iv) $H$ has finitely many orbits on $G/P$.

The assumption of the corollary holds if $H$ and $G$ are complex reductive groups.
Some necessary conditions for finite multiplicities

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Let \( P \subset G \) be a parabolic subgroup defined over \( \mathbb{R} \). Suppose that for all but finitely many orbits of \( H \) on \( G/P \), the set of real points is non-empty and orientable. Then the following are equivalent.

- (i) \( H \) is \( \overline{O}_P \)-spherical.
- (ii) Every \( \pi \in \mathcal{M}_{\overline{O}_P}(G) \) has finite multiplicities in \( S(G/H) \).
- (iii) \( H \) has finitely many orbits on \( G/P \).
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The assumption of the corollary holds if $H$ and $G$ are complex reductive groups. In general however, the finiteness of $|H \backslash G/P|$ is not necessary, but the finiteness of $|H \backslash G/P|$ is not sufficient for finite multiplicities.
Corollary

Let $P \subset G$ be a parabolic subgroup defined over $\mathbb{R}$. Suppose that for all but finitely many orbits of $H$ on $G/P$, the set of real points is non-empty and orientable. Then the following are equivalent.

(i) $H$ is $OP$-spherical.

(ii) Every $\pi \in \mathcal{M}_{OP}(G)$ has finite multiplicities in $S(G/H)$.

(iii) $H$ has finitely many orbits on $G/P$.

(iv) $H$ has finitely many orbits on $G/P$.

The assumption of the corollary holds if $H$ and $G$ are complex reductive groups. In general however, the finiteness of $|H \backslash G/P|$ is not necessary, but the finiteness of $|H \backslash G/P|$ is not sufficient for finite multiplicities. Branching multiplicities for degenerate principal series were computed in various cases by Frahm-Orsted-Oshima, and Kobayashi. Kobayashi: Conditions for bounded multiplicities in terms of distinction w.r. to symmetric $G' \subset G$. 
Example (I. Karshon, related to Howe correspondance in type II)

\( G := \text{Sp}(V \otimes W \oplus V^* \otimes W^*) \), \( H := \text{GL}(V) \times \text{GL}(W) \hookrightarrow G \).

Then \( G / B_H \) is \( O_{\text{min}} \)-spherical.
Further Examples

Example (I. Karshon, related to Howe correspondence in type II)

\[ \mathbf{G} := \text{Sp}(V \otimes W \oplus V^* \otimes W^*), \quad \mathbf{H} := \text{GL}(V) \times \text{GL}(W) \hookrightarrow \mathbf{G}. \]
Then \( \mathbf{G} / B_H \) is \( O_{\text{min}} \)-spherical.

Example (D. Panyushev, strict inequality)

\[ \mathbf{G} := \text{Sp}_{2n}, \quad \mathbf{P} = \text{LU} \subseteq \mathbf{G} \text{- maximal parabolic subgroup with } U \cong \text{Heisenberg group}, \quad \mathbf{O} := O_{\text{min}}. \]
Then \( \dim \mathbf{O} = 2n \), while \( \dim \mathbf{O} \cap p^\perp = 1 \).
Thus \( \dim \mu_{-1}^{-1}(\mathbf{O}) < \dim \mathbf{G} / \mathbf{P} + \dim \mathbf{O} / 2. \)
Step 1 of the proof: Reduction to distributions

**Theorem 3 (Aizenbud - G. 2021)**

Let $I \subset \mathcal{U}(g)$ be a two-sided ideal such that $\mathcal{V}(I) \subset \mathcal{N}(g^*)$. Let $X, Y$ be $\mathcal{V}(I)$-spherical $G$-manifolds. Let $S^*(X \times Y)^{\Delta G, I}$ denote the space of $\Delta G$-invariant tempered distributions on $X \times Y$ annihilated by $I$. Then

$$\dim S^*(X \times Y)^{\Delta G, I} < \infty$$
Step 1 of the proof: Reduction to distributions

Theorem 3 (Aizenbud - G. 2021)

Let $I \subset \mathcal{U}(g)$ be a two-sided ideal such that $\mathcal{V}(I) \subset \mathcal{N}(g^*)$. Let $X, Y$ be $\mathcal{V}(I)$-spherical $G$-manifolds. Let $\mathcal{E}$ be an algebraic vector bundle on $X \times Y$. Let $\mathcal{S}^*(X \times Y, \mathcal{E})^{\Delta G, I}$ denote the space of $\Delta G$-invariant tempered $\mathcal{E}$-valued distributions on $X \times Y$ annihilated by $I$. Then

$$\dim \mathcal{S}^*(X \times Y, \mathcal{E})^{\Delta G, I} < \infty$$
Reduction to distributions

**Theorem 3**

Let \( I \subset \mathcal{U}(\mathfrak{g}) \) be a two-sided ideal such that \( \mathcal{V}(I) \subset \mathcal{N}(\mathfrak{g}^*) \). Let \( X, Y \) be \( \mathcal{V}(I) \)-spherical \( G \)-manifolds. Let \( \mathcal{E} \) be an algebraic vector bundle on \( X \times Y \). Let \( S^*(X \times Y, \mathcal{E})^{\Delta G,I} \) denote the space of \( \Delta G \)-invariant tempered \( \mathcal{E} \)-valued distributions on \( X \times Y \) annihilated by \( I \). Then

\[
\dim S^*(X \times Y, \mathcal{E})^{\Delta G,I} < \infty
\]

**Proof of Theorem 2.**

\( \mathcal{E} \subset \mathcal{N}(\mathfrak{g}^*) \), \( X \) is \( \mathcal{E} \)-spherical, \( \sigma \in \mathcal{M}_\mathcal{E} \). Need: \( \dim \text{Hom}_G(S(X), \sigma) < \infty \).

Let \( \mathcal{E} \) be a bundle on \( Y := G/K \) s.t. \( \sigma \mapsto S^*(Y, \mathcal{E}) \). Let \( I := \text{Ann}(\sigma) \). Then \( \mathcal{V}(I) \subset \mathcal{E} \), and

\[
\text{Hom}_G(S(X), \sigma) \hookrightarrow \text{Hom}_G(S(X), S^*(Y, \mathcal{E}))^{I} \hookrightarrow S^*(X \times Y, \mathbb{C} \boxtimes \mathcal{E})^{\Delta G,I}
\]
Main technique: D-modules

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Let $I \subset \mathcal{U}(g)$ be a two-sided ideal such that $\mathcal{V}(I) \subset \mathcal{N}(g^*)$. Let $X, Y$ be $\mathcal{V}(I)$-spherical $G$-manifolds. Let $E$ be an algebraic vector bundle on $X \times Y$. Let $\mathcal{S}^*(X \times Y, E)^{\Delta G, I}$ denote the space of $\Delta G$-invariant tempered $E$-valued distributions on $X \times Y$ annihilated by $I$. Then

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- $D_X :=$ sheaf of algebraic differential operators. $\text{Gr } D_X \cong O(T^*X)$.
- For a fin.gen. sheaf $M$ of $D_X$-modules, $\text{SingS}(M) := \text{Supp } \text{Gr}(M) \subset T^*X$.
- Bernstein: if $M \neq 0$ then $\dim \text{SingS}(M) \geq \dim X$.
- $M$ is called holonomic if $\dim \text{SingS}(M) = \dim X$. 

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**Theorem (Bernstein-Kashiwara)**

*For any holonomic $M$, $\dim \text{Hom}_{D_X}(M, S^*(X)) < \infty$.***
Theorem 3

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- Bernstein-Kashiwara: \( \forall \) holonomic \( M \), \( \dim \text{Hom}_{D_X}(M, S^*(X)) < \infty \).

Lemma

Let \( \mathcal{E} \subset \mathcal{N}(\mathfrak{g}^*) \) and let \( X, Y \) be \( \mathcal{E} \)-spherical \( G \)-manifolds. Then

\[
\dim \mu_{X \times Y}^{-1}((\mathcal{E} \times \mathcal{E}) \cap (\Delta \mathfrak{g})^\perp) \leq \dim X + \dim Y
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Lemma
Let $\Xi \subset \mathcal{N}(g^*)$ and let $X, Y$ be $\Xi$-spherical $G$-manifolds. Then
\[
\dim \mu_{X \times Y}^{-1}((\Xi \times \Xi) \cap (\Delta g)^\perp) \leq \dim X + \dim Y
\]

Proof of Theorem 3.

$M := D_{X \times Y}$-module with $S^*(X \times Y)^{\Delta G, I} \hookrightarrow \text{Hom}(M, S^*(X, Y))$.
By the lemma, $M$ is holonomic.
Proof of the geometric lemma

Lemma

Let $\Xi \subset \mathcal{N}(g^*)$ and let $X, Y$ be $\Xi$-spherical $G$-manifolds. Then

$$\dim \mu_{X \times Y}^{-1}((\Xi \times \Xi) \cap (\Delta g) \perp) \leq \dim X + \dim Y$$

Proof.

$\forall$ orbit $O \subset \Xi$ we have

$$\dim \mu_{X \times Y}^{-1}((O \times O) \cap (\Delta g) \perp) = \dim \mu_X^{-1}(O) + \dim \mu_Y^{-1}(O) - \dim O \leq \dim X + \dim O/2 + \dim Y + \dim O/2 - \dim O = \dim X + \dim Y$$
Open questions

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Conjecture: Theorem 2 holds over non-archimedean fields as well.

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Dmitry Gourevitch
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