Hypergeometric orthogonal polynomials of Jacobi type

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Branching Problems for Representations of Real, P-Adic and Adelic Groups, Kelowna
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Equivalence: $P_n(x) \sim P_n(\lambda x)$, $P_n(x) \sim e_n P_n(x)$, $e_n \in \mathbb{C}^{\times}$.

Theorem (Bernstein-G.-Sahi '24)

There exist only five families of Jacobi type (up to $\sim$): Jacobi, Laguerre, Bessel, and two families E_n, F_n obtained from Lommel polynomials.

Definitions of the families

$$ {}_iF_j(\underline{a};\underline{b};x) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_i)_k}{(b_1)_k \cdots (b_j)_k k!} x^k, \quad (c)_k := c(c+1)\cdots(c+k-1), $$

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$$ (D(D-1/2) - x(D-n)(D-n-c+1)(D+n+c)(D+n+1))E_n^{(c)} = 0 $$

$$ (D(D+1/2) - x(D-n)(D-n-c+1)(D+n+c+1)(D+n+2))F_n^{(c)} = 0 $$

where $D := x\frac{d}{d_x}$

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- $h_{2n}^{(c)} = (-1)^n E_n^{(c)}(-x^2), \quad h_{2n+1}^{(c)} = (-1)^n x F_n^{(c)}(-x^2),$
- $h_{n+1}^{(c)} = 2(c+n)x h_n^{(c)} - h_{n-1}^{(c)}, \quad h_0 = 1,\ h_{-1} = 0.$

Theorem (Bernstein-G.-Sahi '24)

If $\{P_n\}_{n=0}^{\infty}$ *is a quasi-orthogonal family with*

$$\frac{c(n,k+1)}{c(n,k)} = \frac{p(u,s)}{q(u,s)} \text{ for some } p, q \in \mathbb{C}[u,s]$$

then there exists $g \in \mathbb{C}[u,s]$ *and a family* $\{Q_n\}_{n=0}^{\infty}$ *such that*

$$P_n = g(n, x\partial_x) Q_n \quad \forall n,$$

and $\{Q_n\}$ *is either Jacobi, or Laguerre, or Bessel, or*
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Example

1. $P_n = {}_3F_2(-n, n+1, cn + \frac{c+3}{2}; 3/2, cn + \frac{c+1}{2}; x)$,
   $Q_n = Jacobi(1, 3/2)$.
2. Families not of the form ${}_iF_j$.

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Question

1. *Open questions: 1. Maybe* Q_n *is always of Jacobi type.*
2. *Given a* Q_n *of Jacobi type, what* P_n *are possible?*

Proof Ingredients

- Gauss-Favard: $\{P_n\}$ monic quasi-orthogonal $\iff \exists\{\alpha_n\}, \{\beta_n\} \in \mathbb{C}^{\mathbb{N}}$

$$\text{s.t. } xP_n = P_{n+1} + \alpha_n P_n + \beta_n P_{n-1}, \quad \beta_n \neq 0 \text{ for all } n \geq 1; \quad (1)$$

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- Key lemma: If $\{P_n\}$ of Jacobi type then $\alpha_n, \beta_n \in \mathbb{C}(n)$ (almost), and $c(n,k)$ generate an A-module that is 1-dimensional over $\mathbb{C}(u,s)$.

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- Finish, using clever but elementary considerations.

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Askey-Wilson, 1979: A characterization theorem that leads to new orthogonal polynomials is usually interesting, one that says the classical polynomials are the only polynomials with a given property is usually much less interesting and if it keeps people from looking for new polynomials it is harmful.

Fortunately, ours found new polynomials, and leaves open questions.

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- q-Jacobi type - in progress.
- Askey-Wilson type.

Askey-Wilson, 1979: A characterization theorem that leads to new orthogonal polynomials is usually interesting, one that says the classical polynomials are the only polynomials with a given property is usually much less interesting and if it keeps people from looking for new polynomials it is harmful.

Fortunately, ours found new polynomials, and leaves open questions.

Thank you for your attention!