Hypergeometric orthogonal polynomials of Jacobi type

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Orthogonal families of hypergeometric polynomials

- Let $\{P_n\}_{n=0}^{\infty}$ be a family of polynomials in $\mathbb{C}[x]$ with $\deg P_n = n$.
- $\{P_n\}$ is a *quasi-orthogonal* family if there exists a linear functional $M : \mathbb{C}[x] \to \mathbb{C}$ s.t. $M(P_i P_j) = 0 \iff i \neq j$.
- Write $P_n = \sum_{k=0}^{n} c(n,k)x^k$. Say P_n is *of Jacobi type* if it is quasi-orthogonal and $\exists$ polynomials $p(u,s)$, $q(s)$ s.t.

$$\frac{c(n,k+1)}{c(n,k)} = \frac{p(n,k)}{q(k)}$$

Equivalence: $P_n(x) \sim P_n(\lambda x)$, $P_n(x) \sim e_n P_n(x)$, $e_n \in \mathbb{C}^{\times}$.

Theorem (Bernstein-G.-Sahi '24)

There exist only five families of Jacobi type (up to $\sim$): Jacobi, Laguerre, Bessel, and two families E_n, F_n obtained from Lommel polynomials.

Definitions of the families

$$ {}_iF_j(\underline{a};\underline{b};x) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_i)_k}{(b_1)_k \cdots (b_j)_k k!} x^k, \quad (c)_k := c(c+1)\cdots(c+k-1), $$

If $a_1 = -n$ the infinite series truncates to a polynomial of degree $\leq n$.

1. Jacobi:${}_2F_1(-n, n+a; b; x)$ measure:$(1-x)^{b-1}(1+x)^{a-b}dx$ on $[0,1]$
   $p(u,s) = (s-u)(s+u+a), \quad q(s) = (s+1)(s+b)$.
2. Bessel:${}_2F_0(-n, n+a;;x)$
3. Laguerre: ${}_1F_1(-n; b; x)$ measure: $x^{b-1}\exp(-x)dx$ on $(0,\infty)$
4. $E_n^{(c)}$: ${}_4F_1(-n, -n-c+1, n+c, n+1; 1/2; x)$
5. $F_n^{(c)}$: ${}_4F_1(-n, -n-c+1, n+c+1, n+2; 3/2; x)$

Measures of E_n, F_n are discrete measures defined using zeros of (modified) Bessel functions.

$$ (D(D-1/2) - x(D-n)(D-n-c+1)(D+n+c)(D+n+1))E_n^{(c)} = 0 $$

$$ (D(D+1/2) - x(D-n)(D-n-c+1)(D+n+c+1)(D+n+2))F_n^{(c)} = 0 $$

where $D := x\frac{d}{d_x}$

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- $h_{2n}^{(c)} = (-1)^n E_n^{(c)}(-x^2), \quad h_{2n+1}^{(c)} = (-1)^n x F_n^{(c)}(-x^2),$
- $h_{n+1}^{(c)} = 2(c+n)x h_n^{(c)} - h_{n-1}^{(c)}, \quad h_0 = 1,\ h_{-1} = 0.$

Theorem (Bernstein-G.-Sahi '24)

If $\{P_n\}_{n=0}^{\infty}$ *is a quasi-orthogonal family with*

$$\frac{c(n,k+1)}{c(n,k)} = \frac{p(u,s)}{q(u,s)} \text{ for some } p,q \in \mathbb{C}[u,s]$$

then there exists $g \in \mathbb{C}[u,s]$ *and a family* $\{Q_n\}_{n=0}^{\infty}$ *such that*

$$P_n = g(n, x\partial_x)Q_n \quad \forall n,$$

and $\{Q_n\}$ *is either Jacobi, or Laguerre, or Bessel, or*
$Q_n(x) = {}_4F_1(-n, -n+d, n+a, n+c; b; x)$
for some scalars $a, b, c, d \in \mathbb{C}$.

Example

1. $P_n = {}_3F_2(-n, n+1, cn + \frac{c+3}{2}; 3/2, cn + \frac{c+1}{2}; x)$,
   $Q_n = Jacobi(1, 3/2)$.
2. Families not of the form ${}_iF_j$.

Theorem (Bernstein-G.-Sahi '24)

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Question

1. *Open questions: 1. Maybe* Q_n *is always of Jacobi type.*
2. *Given a* Q_n *of Jacobi type, what* P_n *are possible?*

Proof Ingredients

- Gauss-Favard: $\{P_n\}$ monic quasi-orthogonal $\iff \exists\{\alpha_n\},\{\beta_n\}\in\mathbb{C}^{\mathbb{N}}$

$$\text{s.t. } xP_n = P_{n+1}+\alpha_n P_n+\beta_n P_{n-1}, \quad \beta_n\neq 0 \text{ for all } n\geq 1; \qquad (1)$$

- Algebra $A=\mathbb{C}(u,s)\langle U^{\pm 1},S^{\pm 1}\rangle$ with $Uu=u+1, Ss=s+1$.
- Key lemma: If $\{P_n\}$ of Jacobi type then $\alpha_n,\beta_n\in\mathbb{C}(n)$ (almost), and $c(n,k)$ generate an A-module that is 1-dimensional over $\mathbb{C}(u,s)$.
- Ore (1930s): models for all 1-dimensional A-modules using products of functions of the form $\Gamma(ku+ls)$, $k,l\in\mathbb{Z}$ and $\exp(au+bs)$.
- Elementary analysis of poles using these models and (2):

$$\Phi(u,s-1)=\Phi(u+1,s)+\alpha(u)\Phi(u,s)+\beta(u)\Phi(u-1,s),$$

$$\text{where } \Phi = g\exp(au+bs)\prod\Gamma(k_iu+l_is+c_i),\ g\in\mathbb{C}(u,s)\Rightarrow$$

$$(k_i,l_i)\in\{(\pm1,1),(0,-1),(\pm1,0)\}\ \forall i\Rightarrow$$

$$\Phi(u,s+1)=\frac{p(s-u)r(s+u)Sg}{w(s)g}\Phi(u,s)$$

Proof Ingredients

- Gauss-Favard:$\{P_n\}$ monic quasi-orthogonal $\iff \exists \{\alpha_n\}, \{\beta_n\} \in \mathbb{C}^{\mathbb{N}}$

$$\text{s.t. } xP_n = P_{n+1} + \alpha_n P_n + \beta_n P_{n-1}, \quad \beta_n \neq 0 \text{ for all } n \geq 1; \qquad (2)$$

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$$\text{where } \Phi = g \exp(au + bs) \prod \Gamma(k_i u + l_i s + c_i), \ g \in \mathbb{C}(u,s) \Rightarrow$$

$$(k_i, l_i) \in \{(\pm 1, 1), (0, -1), (\pm 1, 0)\} \ \forall i \Rightarrow$$

$$\Phi(u, s+1) = \frac{p(s-u)r(s+u)Sg}{w(s)g}\Phi(u,s)$$

- Finish, using clever but elementary considerations.

Future plans

- Wilson type polynomials - in progress.
- q-Jacobi type - in progress.
- Askey-Wilson type.

Askey-Wilson, 1979: A characterization theorem that leads to new orthogonal polynomials is usually interesting, one that says the classical polynomials are the only polynomials with a given property is usually much less interesting and if it keeps people from looking for new polynomials it is harmful.

Fortunately, ours found new polynomials, and leaves open questions.

Thank you for your attention!