EXERCISE 1 IN D-MODULES I

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A remark on different kinds of problems. In all my home assignments I will use the following system. The problems without marking are just exercises. You have to convince yourself that you can do them but it is not necessary to write them down (if you have difficulties with one of these problems ask me). The problems marked by (P) you should hand in for grading. The sign (*) marks more difficult problems. The sign (☐) marks more challenging and more interesting problems which are related to some interesting subjects. They are not always related to the course material, but I definitely advise you to think about these problems.

(1) (a) Prove that the algebra $\mathcal{D}_n$ of differential operators on k^n with polynomial coefficients is generated by $x_1, \dots x_n, \partial_1, \dots, \partial_n$ with relations $[x_i, x_j] = [\partial_i, \partial_j] = 0$, $[\partial_i, x_j] = \delta_{ij}$.
  (b) Prove that $x^\alpha \partial^\beta$ form a basis for $\mathcal{D}_n$.

(2) Show that the polynomials $\mathcal{D}_n$-module $k[x_1, \dots, x_n]$ is simple, i.e. has no non-zero proper submodules.

(3) (P)
  (a) Show that the center of $\mathcal{D}_n$ is k.
  (b) Show that $\mathcal{D}_n$ has no two-sided ideals.

(4) ☐ For this problem suppose that the field k has positive characteristic.
  (a) Show that there exists a $\mathcal{D}_n$-module which is finite-dimensional as a k-vector space.
  (b) Show that all simple $\mathcal{D}_n$-modules are finite-dimensional over k.
  (c) Assume that k is algebraically closed and give a classification of all simple $\mathcal{D}_n$-modules.

(5) Let A be an algebra with a fixed good filtration. Let M be an A-module. Prove that
  (a) A filtration F^i on M is good if and only if $Gr_F(M)$ is finitely generated over Gr_A.
  (b) M admits a good filtration if and only if M is finite dimensional.
  (c) If F^i, Φ^i are filtrations and F^i is good then there exists l such that $F^i M \subset \Phi^{i+l} M$.
  (d) Assume that $A = \mathcal{D}_n$ and M is finitely generated. Show that $e(M)$ and $d(M)$ do not depend on the choice of a good filtration.

(6) Let $F^i := \mathcal{D}_n^{\leq i} = \mathrm{Span}\{x^\alpha \partial^\beta \text{ with } |\alpha| + |\beta| \leq i\}$ denote the Bernstein (a.k.a. arithmetic) filtration on $\mathcal{D}_n$. Then $Gr_F(\mathcal{D}_n) \simeq k[x_1, \dots, x_{2n}]$.

(7) Let $f : \mathbb{N} \to \mathbb{Z}$ be an integer sequence. Show that the following are equivalent:
  (a) f is eventually polynomial of degree $\leq d$.
  (b) Δf is eventually polynomial of degree $\leq d - 1$.
  (c) $f(j) \sim \sum_{i=0}^{d} e_i \binom{j}{i}$ where $e_i \in \mathbb{Z}$.

(8) (*) Suppose that k is uncountable and algebraically closed, and let V be a k-vector space of countable dimension. Show that any linear operator $T : V \to V$ has a spectrum, i.e. that $T - \lambda Id$ is not invertible for some $\lambda \in k$.

URL: http://www.wisdom.weizmann.ac.il/~dimagur/Dmod1.html

Date: October 29, 2015.