EXERCISE 10 IN D-MODULES I: DERIVED CATEGORIES

SHACHAR CARMELI

For an abelian category $\mathcal{C}$, let $K(\mathcal{C})$ denote the homotopy category of $\mathcal{C}$ and $D(\mathcal{C})$ the derived category. Let $D^+(\mathcal{C})$, $D^-(\mathcal{C})$, $D^b(\mathcal{C})$ be the subcategories of complexes bounded from below, above and from both sides respectively.

(1) (*) Prove that in $D^+(Ab)$ every object is isomorphic to a complex with zero differentials.

(2) (P) Prove that if $I \in K^+(\mathcal{C})$ is a complex of injective objects, and $M \in K^+(\mathcal{C})$ is acyclic, then $Hom_{K(C)}(M, I) = 0$. Deduce that for every complex A, $Hom_{D^+(\mathcal{C})}(A, I) \cong Hom_{K^+(\mathcal{C})}(A, I)$.

(3) Recall that $RHom(A, B) := Hom(A, I)$ where $B \cong I$ in $D^+(\mathcal{C})$ and I is a complex of injectives. Alternatively, if we work in $D^-(\mathcal{C})$ it is $Hom(P, A)$ where $A \cong P$ and P is a complex of projectives. Prove that $Hom_{D^+(\mathcal{C})}(A, B) = H^0(RHom(A, B))$, and similarly for $D^-(\mathcal{C})$.

(4) (*) Find an example of a complex which is not quasi-isomorphic to a complex with zero-differentials.

(5) (P) Let k be a field. Let $\mathcal{C} = Mod(k[x, y])$. Let A be the complex

$$k[x, y] \xrightarrow{\cdot x^2} k[x, y] \to k[x, y]/(x)$$

and B be the complex

$$k[x, y]/(y^2 - x) \xrightarrow{\cdot y^2} k[x, y]/(y^2 - x) \to k[x, y]/(x, y).$$

Compute the cohomologies of $RHom(A, B)$ and $A \overset{L}{\otimes} B$.

URL: http://www.wisdom.weizmann.ac.il/~dimagur/Dmod1.html

Date: January 18, 2021.