EXERCISE 11 IN D-MODULES I

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(1) (P) A corrected exercise on derived categories: Let k be a field. Let $\mathcal{C} = Mod(k[x,y])$. Let A be the complex
$$k[x,y] \xrightarrow{\cdot x^2} k[x,y] \to k[x,y]/x$$
and B be the complex
$$k[x,y]/(y^2-x) \xrightarrow{\cdot y^2} k[x,y]/(y^2-x) \to k[x,y]/(x,y).$$
Compute the cohomologies of $RHom(A,B)$ and $A \overset{L}{\otimes} B$.

(2) (P) Let M be a smooth $\mathcal{D}_X$-module. Prove that $\mathbb{D}M \cong \mathcal{H}om_{\mathcal{O}_X}(M, \omega_X)$ as an $\mathcal{O}_X$-module.

(3) Show that for any $\mathcal{F}, \mathcal{H} \in D^b_{coh}(\mathcal{M}(\mathcal{D}_X))$, $\mathcal{H}om(\mathcal{F},\mathcal{H}) \cong \mathbb{D}(\mathcal{F}) \otimes^L \mathcal{H}$.

(4) Let $p : X \to pt$, and let $\mathcal{F}, \mathcal{H} \in D^b_{Hol}(\mathcal{M}(\mathcal{D}_X))$. Show that
$$Hom_{\mathcal{D}_X}(\mathcal{F},\mathcal{H}) \cong H^0(p_*(\mathbb{D}\mathcal{H} \otimes^! \mathcal{F})).$$
Hint. Since $Hom_{\mathcal{D}_X}(\mathcal{F},\mathcal{H}) \cong H^0(R\Gamma(R\mathcal{H}om(\mathcal{F},\mathcal{H})))$, and $R\mathcal{H}om(\mathcal{F},\mathcal{H}) \cong \mathbb{D}\mathcal{F} \otimes^L_{\mathcal{D}_X} \mathcal{H}[-\dim X]$, it is enough to show that $\mathbb{D}\mathcal{F} \otimes^L_{\mathcal{D}_X} \mathcal{H}[-\dim X] \cong (\mathbb{D}\mathcal{F} \otimes^! \mathcal{H}) \otimes^L_{\mathcal{D}_X} \mathcal{O}_X$.

(5) Show that $\mathcal{F} \otimes^! \mathcal{O}_X \cong \mathcal{F}$.

(6) (P) Radon transform: let V be a vector space over $\mathbb{K}$, $X = \mathbb{P}(V)$, and $X' = \mathbb{P}(V^*)$. For any $y \in X'$ denote by $H_y \subset X$ the hyperplane on which y vanishes. Let $I = \{x \in X, y \in X' \mid x \in H_y\}$. Let $U := X \times X' \setminus I$ and let j denote the embedding $j : U \subset X \times X'$.

Define $\tilde{R} : D^b(\mathcal{M}(\mathcal{D}_X)) \to D^b(\mathcal{M}(\mathcal{D}'_X))$ by the kernel $j_*\mathcal{O}_U$.

Show that the functor $\tilde{R}$ is an equivalence of categories, with pseudo-inverse given by $j_!\mathcal{O}_U$.

Hint: Show that $K := j_*\mathcal{O}_U * j_!\mathcal{O}_U \cong (\Delta_X)_*\mathcal{O}_X$ in two steps:
1. $\forall x \neq y \in X$, $i^!_{(x,y)}(K) = 0$ 2. $\Delta^!_X K \cong \mathcal{O}_X$.

(7) (P) Let $\mathbb{K} := \mathbb{C}$, and $j : \mathbb{A}^1 \setminus \{0\} \subset \mathbb{A}^1$. Compute $j_{!*}\mathcal{F}$ and $Cone(\varphi_{\mathcal{F}})$ if $\mathcal{F}$ is the $\mathcal{D}(\mathbb{A}^1 \setminus \{0\})$-module generated by the function f on $\mathbb{R}_{>0}$ where:
- (a) $f(x) = x^\lambda$ for some $\lambda \in \mathbb{C}$
- (b) $f(x) = \log x$

(8) This optional problem illustrates non-exactness of Γ for $\mathcal{O}$-modules. Let $X := \mathbb{A}^2 \setminus \{0,0\}$ and $Z := \mathbb{A}^1 \setminus \{0,0\}$. Note that Z is affine while X is not, and $\mathcal{O}_X(X) = \mathbb{K}[x,y]$, $\mathcal{O}_Z(Z) = \mathbb{K}[x,x^{-1}]$.
- (a) Show that the map $\varphi : \mathcal{O}_X \to i_\bullet\mathcal{O}_Z$ is onto, while $\Gamma(\varphi)$ is not onto.
- (b) Find the kernel $K := \mathrm{Ker}\varphi$.
- (c) Compute $H^1(X,K)$ using the Chech resolution with respect to X_x and X_y - the complements to $x = 0$ and to $y = 0$ respectively.
- (d) Compute the boundary map $\Gamma(\mathcal{O}_Z) \to H^1(X,K)$.

URL: http://www.wisdom.weizmann.ac.il/~dimagur/DmodI_3.html

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