EXERCISE 2 IN D-MODULES I

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A remark on different kinds of problems. In all my home assignments I will use the following system. The problems without marking are just exercises. You have to convince yourself that you can do them but it is not necessary to write them down (if you have difficulties with one of these problems ask me). The problems marked by (P) you should hand in for grading. The sign (*) marks more difficult problems. The sign (□) marks more challenging and more interesting problems which are related to some interesting subjects. They are not always related to the course material, but I definitely advise you to think about these problems.

(1) (P) Let M be a $\mathcal{D}_n$-module and let F^iM be a filtration on M, not necessarily good. Suppose that $\dim F^iM < ci^n$ for some constant c and any i. Show that M is finitely generated and, moreover, holonomic.

(2) Let P be a polynomial and recall the $\mathcal{D}_n(k(\lambda))$-module $M_P := M'_P \otimes_{k[\lambda]} k(\lambda)$ defined in the lecture, where $M'_P = \operatorname{Span}\{QP^{\lambda-k} \mid Q \in k[x_1, \ldots, x_n, \lambda]\}$ with the action given by

$$\partial_i(QP^{\lambda-k}) = \partial_i(Q)P^{\lambda-k} + Q(\lambda-k)\partial_i(P)P^{\lambda-k-1}.$$

Construct a (not necessarily good) filtration F^iM_P such that $\dim F^iM_P < ci^n$ for some constant c and any i.

(3) Let P be a real polynomial on $\mathbb{R}^n$. Consider the function $H(x) = \exp(iP(x))$ and denote by F its Fourier transform. Show that F is a holonomic distribution. Later we will see that this implies that F is analytic outside some explicitly described analytic subset.

(4) (P) Let X be a real vector space. Fix a real polynomial P on X. Fix a not very singular distribution ξ on X (e.g. a finite sum of differential operators applied to continuous functions) and define a family of generalized functions $G(\lambda) := P^\lambda f$ for $\operatorname{Re}\lambda >> 0$. Show that if ξ is holonomic then the family extends meromorphically to the whole complex plane. More precisely, show that there exists a differential operator $d \in D(X)[\lambda]$ and a polynomial $b \in C[\lambda]$ such that $d(\lambda)G(\lambda+1) = b(\lambda)G(\lambda)$.

(5) (P)
   (a) Similarly to the previous problem, show that the function $G(\lambda, \mu) = P^\lambda G^\mu \xi$ has a meromorphic continuation in two variables λ, μ.
   (b) Show that the function $b(\lambda, \mu)$ can be chosen to be a product of linear functions.

URL: http://www.wisdom.weizmann.ac.il/~dimagur/Dmod1.html

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