EXERCISE 3 IN D-MODULES I

DMITRY GOUREVITCH AND JOSEPH BERNSTEIN

(1) Exterior products of $\mathcal{D}$-modules. Consider to affine spaces X, Y. Let M be a $\mathcal{D}(X)$-module and N be a $\mathcal{D}(Y)$-module.
   (a) Consider the vector space $M \otimes_k N$ and define on it a structure of a $\mathcal{D}(X \times Y)$-module. This module is called the exterior product of M and N and denoted $M \boxtimes N$.
   (b) Prove that $d(M \boxtimes N) = d(M) + d(N)$ and $e(M \boxtimes N) = e(M)e(N)$.

(2) Tensor product over $\mathcal{O}$. Let M, N be $\mathcal{D}(X)$-modules (where X is an affine space). Consider the space $M \otimes N := M \otimes_{\mathcal{O}(X)} N$ and define the structure of a $\mathcal{D}(X)$-module by Leibnitz rule.
   (a) Show that $M \otimes N$ is canonically isomorphic to the module $\Delta^0(M \boxtimes N)$, where $\Delta : X \to X \times X$ is the diagonal embedding
   (b) Interpret the analytic meaning of this algebraic operation. Show that the inner tensor product $M \otimes N$ of finitely-generated $\mathcal{D}$-modules is not always finitely generated.

(3) (P) Show that if M, N are holonomic then $M \otimes N$ is also holonomic.

(4) (P) Let M be a left $\mathcal{D}(X)$-module and N be a right $\mathcal{D}(X)$-module. Define a natural structure of a right $\mathcal{D}(X)$-module on $M \otimes N$. Explain the analytic meaning of this construction. Convince yourself that there is no natural tensor product of right $\mathcal{D}(X)$-modules.

(5) (P) Let X be a real vector space. Fix a real polynomial P on X. Fix a not very singular distribution ξ on X (e.g. a finite sum of differential operators applied to continuous functions) and define a family of generalized functions $G(\lambda) := P^\lambda f$ for $\mathrm{Re}\,\lambda >> 0$. Show that if ξ is holonomic then the family extends meromorphically to the whole complex plane. More precisely, show that there exists a differential operator $d \in \mathcal{D}(X)[\lambda]$ and a polynomial $b \in C[\lambda]$ such that $d(\lambda)G(\lambda + 1) = b(\lambda)G(\lambda)$.

(6) (P)
   (a) Similarly to the previous problem, show that the function $G(\lambda, \mu) = P^\lambda G \mu \xi$ has a meromorphic continuation in two variables λ, μ.
   (b) Show that the function $b(\lambda, \mu)$ can be chosen to be a product of linear functions.

(7) (P) Let T be a differential operator with constant coefficients on $\mathbb{R}^n$. Show that there exists a distribution f with $Tf = \delta_0$. Moreover, f can be chosen to be a tempered holonomic distribution.

(8) (P) Let M be a $\mathcal{D}_n$-module generated by a finite subset S. Let $I \subset \mathcal{D}_n$ be the annihilator of S, and let $J \subset k[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$ be the ideal generated by the symbols of the elements of I. Show that the associated variety $AV(M)$ is the zero set of J.

URL: http://www.wisdom.weizmann.ac.il/~dimagur/Dmod1.html

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