# EXERCISE 3 IN D-MODULES I

DMITRY GOUREVITCH AND JOSEPH BERNSTEIN

(1) Exterior products of $\mathcal{D}$-modules. Consider to affine spaces $X, Y$. Let $M$ be a $\mathcal{D}(X)$-module and $N$ be a $\mathcal{D}(Y)$-module.
  (a) Consider the vector space $M \otimes_k N$ and define on it a structure of a $\mathcal{D}(X \times Y)$-module. This module is called the exterior product of $M$ and $N$ and denoted $M \boxtimes N$.
  (b) Prove that
$$d(M \boxtimes N) = d(M) + d(N) \text{ and } e(M \boxtimes N) \leq \binom{d(M) + d(N)}{d(M)} e(M)e(N).$$
(2) Tensor product over $\mathcal{O}$. Let $M, N$ be $\mathcal{D}(X)$-modules (where $X$ is an affine space). Consider the space $M \otimes N := M \otimes_{\mathcal{O}(X)} N$ and define the structure of a $\mathcal{D}(X)$-module by Leibnitz rule.
  (a) Show that $M \otimes N$ is canonically isomorphic to the module $\Delta^0(M \boxtimes N)$, where $\Delta : X \to X \times X$ is the diagonal embedding
  (b) Interpret the analytic meaning of this algebraic operation. Show that the inner tensor product $M \otimes N$ of finitely-generated $\mathcal{D}$-modules is not always finitely generated.
(3) (P) Show that if $M, N$ are holonomic then $M \otimes N$ is also holonomic.
(4) (P) Let $M$ be a left $\mathcal{D}(X)$-module and $N$ be a right $\mathcal{D}(X)$-module. Define a natural structure of a right $\mathcal{D}(X)$-module on $M \otimes N$. Explain the analytic meaning of this construction. Convince yourself that there is no natural tensor product of right $\mathcal{D}(X)$-modules.
(5) (P) Let $X$ be a real vector space. Fix a real polynomial $P$ on $X$. Fix a not very singular distribution $\xi$ on $X$ (e.g. a finite sum of differential operators applied to continuous functions) and define a family of generalized functions $G(\lambda) := P^\lambda \xi$ for $\operatorname{Re} \lambda >> 0$. Show that if $\xi$ is holonomic then the family extends meromorphically to the whole complex plane. More precisely, show that there exists a differential operator $d \in \mathcal{D}(X)[\lambda]$ and a polynomial $b \in C[\lambda]$ such that $d(\lambda)G(\lambda + 1) = b(\lambda)G(\lambda)$.
(6) (P)
  (a) Similarly to the previous problem, show that the function $G(\lambda, \mu) = P^\lambda G^\mu \xi$ has a meromorphic continuation in two variables $\lambda, \mu$.
  (b) (*)Show that the function $b(\lambda, \mu)$ can be chosen to be a product of linear functions.

*URL*: http://www.wisdom.weizmann.ac.il/~dimagur/Dmod1.html

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