EXERCISE 4 IN D-MODULES I

DMITRY GOUREVITCH AND JOSEPH BERNSTEIN

Let A be a Noetherian algebra, $\mathcal{M}(A)$ denote the category of A-modules and $\mathcal{M}^f(A)$ denote the subcategory of finitely-generated A-modules.

(1) Show that $P \in \mathcal{M}^f(A)$ is projective in $\mathcal{M}^f(A)$ if and only if it is projective in $\mathcal{M}(A)$. Show that the homological dimensions of $\mathcal{M}^f(A)$ and $\mathcal{M}(A)$ are equal.
(2) Show that a module M is finitely-generated if and only if for any system of submodules satisfying $\sum M_\alpha = M$ there exists a finite subsystem with this property.

(3) (P) Let $M \in \mathcal{M}^f(A)$. Suppose that M has a projective resolution of length d. Consider the functor $E : N \mapsto Ext^d(M, N)$. Show that there exists a right A-module R such that this functor is isomorphic to a functor T_R defined by $T_R(N) := R \otimes_A N$. Show that the module R is defined uniquely up to canonical isomorphism.

(4) (P) Let $\mathcal{C}$ be an abelian category. Let $\Pi \in \mathcal{C}$ be a projective object. Suppose that arbitrary direct powers of Π are defined, and for any object $M \in \mathcal{C}$ there exist a power of Π and an epimorphism $\Pi^\alpha \twoheadrightarrow M$. Show that the $\mathcal{C}$ is equivalent to the category of right modules over the ring $\mathrm{End}(\Pi)$.

Direct limits.

Definition 1. *Let I be a partially ordered set. We will consider it as a category with one morphism $i \to j$ if $i \leq j$, and no morphisms otherwise. An I-**system** of objects in a category $\mathcal{M}$ is a functor $I \to \mathcal{M}$. I is called **directed** if for any $i, j \in I$ there exists $l \in I$ with $i, j \leq l$. The **direct limit (or a colimit)** $\varinjlim F$ of a system $F : I \to \mathcal{M}$ is an object $A \in \mathcal{M}$ and an isomorphism of the functors $\mathrm{Hom}(A, \cdot)$ and the functor G that sends every object $B \in \mathcal{M}$ to the set of natural transformations between F and the constant functor $I \to \mathcal{M}$ that sends every object to B and every map to identity. Sometimes $\varinjlim F$ denotes just the object A.*

(5) Construct colimits in the category of sets and in $\mathcal{M}(A)$.
(6) Show that any $M \in \mathcal{M}(A)$ is a direct limit of a directed system in $\mathcal{M}^f(A)$.
(7) Show that if I is a directed system and $\mathcal{M}$ an abelian category then the functor $F \mapsto \varinjlim F$ is exact.

(8) (P) Show that an A-module M is finitely-generated if and only if the functor $\mathcal{M}(A) \to Ab$ given by $N \mapsto \mathrm{Hom}(M, N)$ commutes with arbitrary directed direct limits. Moreover, show that if $M \in \mathcal{M}^f(A)$ then $Ext^i(M, \cdot)$ commutes with directed direct limits, and $\mathrm{Hom}(M, \cdot)$ commutes with arbitrary direct limits. Do $Ext^i(M, \cdot)$ commute with arbitrary direct limits?

URL: http://www.wisdom.weizmann.ac.il/~dimagur/Dmod1.html

Date: November 26, 2015.