

EXERCISE 4 IN D-MODULES I

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Let A be a Noetherian algebra, $\mathcal{M}(A)$ denote the category of A -modules and $\mathcal{M}^f(A)$ denote the subcategory of finitely-generated A -modules.

- (1) Show that a module M is finitely-generated if and only if for any system of submodules satisfying $\sum M_\alpha = M$ there exists a finite subsystem with this property.
- (2) (P) Let $M \in \mathcal{M}^f(A)$. Suppose that M has a projective resolution of length d . Consider the functor $E : N \mapsto \text{Ext}^d(M, N)$. Show that there exists a right A -module R such that this functor is isomorphic to a functor T_R defined by $T_R(N) := R \otimes_A N$. Show that the module R is defined uniquely up to canonical isomorphism.
- (3) (P) Let \mathcal{C} be an abelian category. Let $\Pi \in \mathcal{C}$ be a projective object. Suppose that arbitrary direct powers of Π are defined, and for any object $M \in \mathcal{C}$ there exist a power of Π and an epimorphism $\Pi^\alpha \rightarrow M$. Show that the \mathcal{C} is equivalent to the category of right modules over the ring $\text{End}(\Pi)$.

Direct limits.

Definition 1. Let I be a partially ordered set. We will consider it as a category with one morphism $i \rightarrow j$ if $i \leq j$, and no morphisms otherwise. An I -**system** of objects in a category \mathcal{M} is a functor $I \rightarrow \mathcal{M}$. I is called **directed** if for any $i, j \in I$ there exists $l \in I$ with $i, j \leq l$. The **direct limit (or a colimit)** $\varinjlim F$ of a system $F : I \rightarrow \mathcal{M}$ is an object $A \in \mathcal{M}$ and an isomorphism of the functors $\text{Hom}(A, \cdot)$ and the functor G that sends every object $B \in \mathcal{M}$ to the set of natural transformations between F and the constant functor $I \rightarrow \mathcal{M}$ that sends every object to B and every map to identity. Sometimes $\varinjlim F$ denotes just the object A .

- (5) Construct colimits in the category of sets and in $\mathcal{M}(A)$.
- (6) Show that any $M \in \mathcal{M}(A)$ is a direct limit of a directed system in $\mathcal{M}^f(A)$.
- (7) Show that if I is a directed system and \mathcal{M} an abelian category then the functor $F \mapsto \varinjlim F$ is exact.
- (8) (P) Show that an A -module M is finitely-generated if and only if the functor $\mathcal{M}(A) \rightarrow \text{Ab}$ given by $N \mapsto \text{Hom}(M, N)$ commutes with arbitrary directed direct limits. Moreover, show that if $M \in \mathcal{M}^f(A)$ then $\text{Ext}^i(M, \cdot)$ commutes with directed direct limits, and $\text{Hom}(M, \cdot)$ commutes with arbitrary direct limits. Do $\text{Ext}^i(M, \cdot)$ commute with arbitrary direct limits?

URL: <http://www.wisdom.weizmann.ac.il/~dimagur/Dmod1.html>