EXERCISE 5 IN D-MODULES I

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(1) Show that the left ideal $\mathcal{D}_1\langle\partial^2, x\partial - 1\rangle$ is not cyclic. Show that the left ideal $\mathcal{D}_3\langle\partial_1, \partial_2, \partial_3\rangle$ equals $\mathcal{D}_3\langle\partial_1, \partial_2 + x_1\partial_3\rangle$, but is not cyclic.
(2) Define $\Phi : \mathcal{D}_1 \oplus \mathcal{D}_1 \to \mathcal{D}_1$ by $\Phi(a, b) := a\partial + bx$. Show that Φ is onto and $\operatorname{Ker}\Phi$ is isomorphic to the ideal $\mathcal{D}_1\langle x^2, \partial x\rangle$. Conclude that this ideal is a projective module.
(3) (P)
   (a) Show that for any $d \in \mathcal{D}_1$, the space of polynomial solutions of d is finite-dimensional.
   (b) Show that the space of solutions of $x_1\partial_2 - x_2\partial_1$ in $k[x_1, x_2]$ is infinite-dimensional.
(4) (P)(*) Show directly (without using homological theorems) that the arithmetic and the geometric functional dimensions of any finitely-generated $\mathcal{D}_n$-module are equal.
(5) (P) Let $M \in \mathcal{M}^f(\mathcal{D}_n)$ and $H := Ext^n(M, \mathcal{D}_n)$. Show that the image of the map $DH \to M$ constructed in class contains all holonomic submodules of M.

URL: http://www.wisdom.weizmann.ac.il/~dimagur/Dmod1.html

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