EXERCISE 5 IN D-MODULES I

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(1) (P) Define $\Phi : \mathcal{D}_1 \oplus \mathcal{D}_1 \to \mathcal{D}_1$ by $\Phi(a, b) := a\partial + bx$. Show that Φ is onto and $\operatorname{Ker}\Phi$ is isomorphic to the ideal $\mathcal{D}_1\langle x^2, \partial x\rangle$. Conclude that this ideal is a projective module.

(2) Let V be a vector space, and $a_1, \dots, a_n : V \to V$ be a regular sequence of commuting linear operators. Then the Koszul complex is acyclic outside 0, and

$$H_0(C) \simeq V/(a_1V + \dots + a_nV).$$

(See the lecture for the definitions of regular sequence and Kozhul complex)

(3) (*) $\operatorname{hd}(\mathcal{M}(R)) = \operatorname{hd}(R)$.

(4) Let R be a ring and $M \in \mathcal{M}^f(R)$ with a good filtration. Then
   (i) for some l there exists a good filtration on R^l and a strict epimorphism $R^l \twoheadrightarrow M$.
   (ii) If $\operatorname{Gr} M$ is free then M is free.

(5) (P) Let $L := k[x, x^{-1}]$, $M := k[x]$ and $C := L/M$. Note that they are all holonomic and consider the exact sequence $0 \to M \to L \to C \to 0$.

Compute the dual D-modules, and describe the dual exact sequence

$$0 \to D(C) \to D(L) \to D(M) \to 0$$

in terms of distributions.

(6) Let $\mathcal{C}$ be an abelian category. Let $\Pi \in \mathcal{C}$ be a projective object. Suppose that arbitrary direct powers of Π are defined, and for any object $M \in \mathcal{C}$ there exist a power of Π and an epimorphism $\Pi^\alpha \twoheadrightarrow M$. Show that the $\mathcal{C}$ is equivalent to the category of right modules over the ring $\operatorname{End}(\Pi)$.

Direct limits.

Definition 1. *Let I be a partially ordered set. We will consider it as a category with one morphism $i \to j$ if $i \le j$, and no morphisms otherwise. An I-**system** of objects in a category $\mathcal{M}$ is a functor $I \to \mathcal{M}$. I is called **directed** if for any $i, j \in I$ there exists $l \in I$ with $i, j \le l$. The **direct limit (or a colimit)** $\varinjlim F$ of a system $F : I \to \mathcal{M}$ is an object $A \in \mathcal{M}$ and an isomorphism of the functors $\operatorname{Hom}(A, \cdot)$ and the functor G that sends every object $B \in \mathcal{M}$ to the set of natural transformations between F and the constant functor $I \to \mathcal{M}$ that sends every object to B and every map to identity. Sometimes $\varinjlim F$ denotes just the object A.*

Let A be a Noetherian algebra, $\mathcal{M}(A)$ denote the category of A-modules and $\mathcal{M}^f(A)$ denote the subcategory of finitely-generated A-modules.

(3) Show that a module M is finitely-generated if and only if for any system of submodules satisfying $\sum M_\alpha = M$ there exists a finite subsystem with this property.
(4) Construct colimits in the category of sets and in $\mathcal{M}(A)$.
(5) Show that any $M \in \mathcal{M}(A)$ is a direct limit of a directed system in $\mathcal{M}^f(A)$.
(6) Show that if I is a directed system and $\mathcal{M}$ an abelian category then the functor $F \mapsto \varinjlim F$ is exact.

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(7) Show that an A-module M is finitely-generated if and only if the functor $\mathcal{M}(A) \to Ab$ given by $N \mapsto \mathrm{Hom}(M, N)$ commutes with arbitrary directed direct limits. Moreover, show that if $M \in \mathcal{M}^f(A)$ then $Ext^i(M, \cdot)$ commutes with directed direct limits, and $\mathrm{Hom}(M, \cdot)$ commutes with arbitrary direct limits. Do $Ext^i(M, \cdot)$ commute with arbitrary direct limits?

URL: http://www.wisdom.weizmann.ac.il/~dimagur/DmodI_3.html