EXERCISE 6 IN D-MODULES I

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Let X be a smooth algebraic variety and $Z \subset X$ a (Zariski) closed subvariety. Let $i : Z \hookrightarrow X$ denote the embedding.

(1) Let $\mathcal{H} \in \mathcal{M}^R_{coh}(\mathcal{D}_Z)$. Show that $i_0(\mathcal{H})$ is also coherent and
$$SS(i_0(\mathcal{H})) = \{(x, \xi) \in T^*X \,|\, x \in Z \text{ and } (x, p_{X,x}(\xi)) \in SS(\mathcal{H})\},$$
where $p_{X,x} : T^*_x X \to T^*_z X$ denotes the standard projection.

(2) Define $i' : \mathcal{M}^R(\mathcal{D}_X) \to \mathcal{M}^R(\mathcal{D}_Z)$ by $i'(\mathcal{F}) := \mathcal{H}om(\mathcal{D}_{Z\to X}, \mathcal{F})$. Show that
   (a) If X is affine, and $I(Z) \subset \mathcal{O}(X)$ is the ideal of all functions vanishing on Z then
   $$i'M = \{m \in M \,|\, am = 0 \,\forall a \in I(Z)\},$$
   for any right $\mathcal{D}(X)$-module M.
   (b) $i'i_0\mathcal{H}$ is naturally isomorphic to $\mathcal{H}$.
   (c) i_0 is left adjoint to i'.

(3) (P) Let $i : \mathbb{A}^1 \to \mathbb{A}^2$ be the standard embedding. Compute $i_0 M$ for
   (a) $M = K[x]$
   (b) $M = \mathcal{D}_1/\partial^5$
   (c) $M = \mathcal{D}_1/x\partial$
   (d) $M = \mathcal{D}_1/(x^2\partial + 4x + 1)$

(4) Show that if X is affine then the structure of a left $\mathcal{D}(X)$-module on an $\mathcal{O}(X)$-module M is the same as the action of the Lie algebra $\mathcal{T}(X)$ of algebraic vector fields, that satisfies
   (a) $[\langle\xi_1\rangle, \langle\xi_2\rangle = \langle[\xi_1, \xi_2]\rangle$
   (b) $[\langle\xi\rangle, \langle f\rangle] = \langle\xi(f)\rangle$
   (c) $\langle f\xi\rangle = \langle f\rangle \cdot \langle\xi\rangle$,
   where $\xi, \xi_1, \xi_2 \in \mathcal{T}(X)$, $f \in \mathcal{O}(X)$, and $\langle \cdot \rangle$ denotes the action on M.

(5) Let $M \in \mathcal{M}(\mathcal{D}_n)$. Let $\varphi : M \to M[x_n^{-1}]$ be the natural map. Show that $\ker\varphi$ and $coker\varphi$ are supported on the hyperplane $\{x_n = 0\}$.

URL: http://www.wisdom.weizmann.ac.il/~dimagur/Dmod1.html

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