EXERCISE 6 IN D-MODULES I

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Let X be a smooth algebraic variety and $Z \subset X$ a (Zariski) closed subvariety. Let $i : Z \hookrightarrow X$ denote the embedding.

(1) (P) Show that for any smooth affine variety X, $\mathcal{D}^{\leq 0}(X) = \mathcal{O}(X)$, and
$$\mathcal{D}^{\leq 1}(X)/\mathcal{O}(X) \cong \mathrm{Der}\mathcal{O}(X).$$
(2) (*)
  (a) If $X = \left\{\sum_i x_i^2 = 0\right\}$ then $\mathcal{D}(X)$ is Noetherian but not generated by $\mathcal{D}^{\leq 1}(X)$.
  (b) If $X = \left\{\sum_i x_i^3 = 0\right\}$ then $\mathcal{D}(X)$ is not Noetherian.

(3) (P) Let $\mathcal{H} \in \mathcal{M}^R_{coh}(\mathcal{D}_Z)$. Show that $i_0(\mathcal{H})$ is also coherent and
$$SS(i_0(\mathcal{H})) = \{(x,\xi) \in T^*X \mid x \in Z \text{ and } (x, p_{X,x}(\xi)) \in SS(\mathcal{H})\},$$
where $p_{X,x} : T^*_x X \to T^*_x Z$ denotes the standard projection.

(4) (P) Let $i : \mathbb{A}^1 \to \mathbb{A}^2$ be the standard embedding. Compute $i_0 M$ for
  (a) $M = K[x]$
  (b) $M = \mathcal{D}_1/\partial^5$
  (c) $M = \mathcal{D}_1/x\partial$
  (d) $M = \mathcal{D}_1/(x^2\partial + 4x + 1)$

(5) Show that if X is affine then the structure of a left $\mathcal{D}(X)$-module on an $\mathcal{O}(X)$-module M is the same as the action of the Lie algebra $\mathcal{T}(X)$ of algebraic vector fields, that satisfies
  (a) $[\langle\xi_1\rangle, \langle\xi_2\rangle = \langle[\xi_1, \xi_2]\rangle$
  (b) $[\langle\xi\rangle, \langle f\rangle] = \langle\xi(f)\rangle$
  (c) $\langle f\xi\rangle = \langle f\rangle \cdot \langle\xi\rangle$,

  where $\xi, \xi_1, \xi_2 \in \mathcal{T}(X)$, $f \in \mathcal{O}(X)$, and $\langle \cdot \rangle$ denotes the action on M.

(6) Let $M \in \mathcal{M}(\mathcal{D}_n)$. Let $\varphi : M \to M[x_n^{-1}]$ be the natural map. Show that $\ker\varphi$ and $coker\varphi$ are supported on the hyperplane $\{x_n = 0\}$.

(7) The module of top differential forms Ω_X^{top} with the action $\xi\alpha := -Lie_\xi\alpha$ (Lie derivative) is a right $\mathcal{D}(X)$-module. Moreover, $M \mapsto M \otimes_{\mathcal{O}_X} \Omega_X^{top}$ defines an equivalence of categories $\mathcal{M}(\mathcal{D}(X)) \simeq \mathcal{M}^r(\mathcal{D}(X))$.

URL: http://www.wisdom.weizmann.ac.il/~dimagur/DmodI_3.html

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