EXERCISE 7 IN D-MODULES I: DERIVED CATEGORIES

SHACHAR CARMELI

For an abelian category $\mathcal{C}$, let $K(\mathcal{C})$ denote the homotopy category of $\mathcal{C}$ and $D(\mathcal{C})$ the derived category. Let $D^+(\mathcal{C})$, $D^-(\mathcal{C})$, $D^b(\mathcal{C})$ be the subcategories of complexes bounded from below, above and from both sides respectively. If $f : A \to B$ we denote by $C(f)$ the mapping cone, and by $\pi_f : B \to C(f)$ the inclusion to the second factor.

(1) Prove that $C(\pi_f) \cong A[1]$.
(2) Prove that $H^i(A) \xrightarrow{f_*} H^i(B) \xrightarrow{(\pi_f)_*} H^i(C(f))$ is exact. Deduce that a distinguished triangle induces a long exact sequence on cohomologies.
(3) (P) Prove that $f : A \to B$ is a homotopy equivalence if and only if $C(f)$ is contractible (i.e. isomorphic to 0 in $K(\mathcal{C})$).
(4) (*) Prove that every distinguished triangle in $K(\mathcal{C})$ is isomorphic to a short exact sequence. Deduce once again that a distinguished triangle induces a long exact sequence on cohomologies. Does every short exact sequence is a distinguished triangle in $K(\mathcal{C})$? Prove that in $D(\mathcal{C})$, every short exact sequence is a distinguished triangle.
(5) (P) Prove that for $g : C \to B$ and $f : A \to B$, $g = g' \circ f$ if and only if $\pi_f \circ g = 0$. State and prove a dual statement.
(6) (P) Define the complex $Hom(A, B)$, and prove that $H^0(Hom(A, B)) = Hom_{K(\mathcal{C})}(A, B)$. Prove that $Hom(X, C(f)) = C(Hom(X, f))$. Use this to solve the last exercise much faster!
(7) (*) Prove that in $D(\mathbb{Z})$ every object is isomorphic to a complex with zero-differentials.
(8) (P) Prove that if $I \in K^+(\mathcal{C})$ is a complex of injective objects, and $M \in K^+(\mathcal{C})$ is acyclic, then $Hom_{K(C)}(M, I) = 0$. Deduce that for every complex A, $Hom_{D^+(\mathcal{C})}(A, I) \cong Hom_{K^+(\mathcal{C})}(A, I)$.
(9) Recall that $RHom(A, B) := Hom(A, I)$ where $B \cong I$ in $D^+(\mathcal{C})$ and I is a complex of injectives. Alternatively, if we work in $D^-(\mathcal{C})$ it is $Hom(P, A)$ where $A \cong P$ and P is a complex of projectives. Prove that $Hom_{D^+(\mathcal{C})}(A, B) = H^0(RHom(A, B))$, and similarly for $D^-(\mathcal{C})$.
(10) (P) Prove that for every $A \in D^-(Ab)$, $RHom(RHom(A, \mathbb{Z}), \mathbb{Z}) \cong A$. Thus, if $\mathbb{D}(A) := RHom(A, \mathbb{Z})$, then

$$Hom_{D^-(Ab)}(A, B) \cong Hom_{D^-(Ab)}(\mathbb{D}(B), \mathbb{D}(A)).$$

(11) (*) Find an example of a complex which is not quasi-isomorphic to a complex with zero-differentials.
(12) (P) Let k be a field. Let $\mathcal{C} = Mod(k[x, y])$. Let A be the complex

$$k[x, y] \xrightarrow{\cdot x} k[x, y] \to k[x, y]/(x^2)$$

and B be the complex

$$k[x, y]/(y^2 - x) \xrightarrow{\cdot y^2} k[x, y]/(y^2 - x) \to k[x, y]/(x, y).$$

Compute the cohomologies of $RHom(A, B)$ and $A \overset{L}{\otimes} B$.

URL: http://www.wisdom.weizmann.ac.il/~dimagur/Dmod1.html

Date: January 26, 2016.