EXERCISE 8 IN D-MODULES I

(1) Any TDO on $\mathbb{P}^n$ is given by the coordinate changes

$$\varphi_{ij}(\partial_k) := \partial_k + \lambda \partial_k(x_i/x_j) \cdot (x_i/x_j)^{-1}.$$

(2) (P) Denote by $\mathcal{D}_{\mathbb{P}^n}(s)$ the sheaf of differential operators on $\mathcal{O}_{\mathbb{P}^n}(s)$. Then the global sections functor $\Gamma : \mathcal{M}(\mathcal{D}_{\mathbb{P}^n}(s)) \to \mathcal{M}(\Gamma(\mathcal{D}_{\mathbb{P}^n}(s)))$ is exact for $s > -n$ and faithful for $s \geq 0$.

(3) Let X be affine. For a closed 1-form λ on X and $\eta \in \tau_X$ define $\varphi_\lambda(\eta) := \eta + \lambda(\eta) \in \mathcal{D}(X)$. Then φ_λ extends (uniquely) to an automorphism of $\mathcal{D}(X)$ as an $\mathcal{O}(X)$-algebra. Moreover, all automorphisms of $\mathcal{D}(X)$ as an $\mathcal{O}(X)$-algebra are obtained in this way.

(4) Show that the space of solutions of the equation $x_1\partial_2 - x_2\partial_1$ in $\mathbb{K}[x_1, x_2]$ is infinite-dimensional.

(5) Let $\nu : \mathbb{A}^1 \to \mathbb{A}^1$ be given by $\nu(x) = x^m$. Let M be the $\mathcal{D}_1$-module generated by δ_0. Show that $\nu^*(M)$ is the $\mathcal{D}_1$-module generated by $\delta_0^{(m-1)}$. We did this in class, and the exercise here is to fill in the details, not for submission. Steps:
   (i) Show that $1 \otimes \delta_0 \in \nu^*(M) = \mathbb{K}[x] \otimes_{\mathbb{K}[y]} M$ (where $y = x^m$) is annihilated by x^m and by $x\partial_x + m$
   (ii) Show that $1 \otimes \delta_0$ generates $\nu^*(M)$. Hint: show by induction that for any natural numbers p, q with $0 \leq q \leq m$ there exist non-zero numbers $c_{pq} \in \mathbb{K}$ such that

$$\partial_x^{mp+q}(1 \otimes \delta) = c_{pq} x^{m-q} \partial_y^{p+1} \delta,$$

   and conclude that $\partial_y^{p+1}\delta = c_{pq}^{-1} \partial_x^{m(p+1)}(1 \otimes \delta)$.
   (iii) Show that the module $\mathcal{D}_1/ < x^m, x\partial_x + m >$ is irreducible, and conclude that is isomorphic to its quotient $\nu^*(M)$.

(6) (P) Let $d \in \mathcal{D}_n$ be an operator of degree 2.
   (a) Show that there exists an automorphism α of $\mathcal{D}_n$ such that $\alpha(d) = \sum_{i=1}^n \lambda_i(x_i^2 + \partial_i^2) + r_i$.
   (b) Show that the $\mathcal{D}_n$-module $\mathcal{D}_n/d\mathcal{D}_n$ has a holonomic quotient.

URL: http://www.wisdom.weizmann.ac.il/~dimagur/DmodI_3.html

Date: December 15, 2020.