EXERCISE 9 IN D-MODULES I: HOMOTOPY CATEGORIES

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For an abelian category $\mathcal{C}$, let $K(\mathcal{C})$ denote the homotopy category of $\mathcal{C}$ and $D(\mathcal{C})$ the derived category. If $f : A \to B$ we denote by $\mathrm{Cone}(f)$ the mapping cone, and by $\pi_f : B \to \mathrm{Cone}(f)$ the inclusion to the second factor.

(1) Prove that $\mathrm{Cone}(\pi_f) \cong A[1]$.
(2) Prove that $H^i(A) \xrightarrow{f_*} H^i(B) \xrightarrow{(\pi_f)_*} H^i(\mathrm{Cone}(f))$ is exact. Deduce that a distinguished triangle induces a long exact sequence on cohomologies.
(3) (P) Prove that $f : A \to B$ is a homotopy equivalence if and only if $\mathrm{Cone}(f)$ is contractible (i.e. isomorphic to 0 in $K(\mathcal{C})$).
(4) Prove that $f : A \to B$ is a quasiisomorphism if and only if $\mathrm{Cone}(f)$ is acyclic.
(5) (*) Prove that every distinguished triangle in $K(\mathcal{C})$ is isomorphic to a short exact sequence. Deduce once again that a distinguished triangle induces a long exact sequence on cohomologies. Is every short exact sequence a distinguished triangle in $K(\mathcal{C})$? Prove that in $D(\mathcal{C})$, every short exact sequence is a distinguished triangle.
(6) (P) Prove that for $g : C \to B$ and $f : A \to B$, $g = f \circ g'$ if and only if $\pi_f \circ g = 0$. State and prove a dual statement.
(7) (P) Define the complex $Hom(A, B)$, and prove that $H^0(Hom(A, B)) = Hom_{K(\mathcal{C})}(A, B)$. Prove that $Hom(X, C(f)) = C(Hom(X, f))$. Use this to solve the last exercise much faster!

URL: http://www.wisdom.weizmann.ac.il/~dimagur/Dmod1.html

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