EXERCISE 1 IN D-MODULES II

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(1) (P) Show that a complex $\mathcal{F}$ of coherent $\mathcal{D}_X$-modules is holonomic if and only if for every open affine $U \subset X$, the complex $p_*(\mathcal{F}|_U)$ has finite-dimensional cohomologies, where p is the projection $p : U \to pt$.

Regular singular D-modules on curves. Let C be a smooth curve and $\overline{C}$ be its (smooth) completion. Let $\mathcal{E}$ be a smooth $\mathcal{D}_C$-module. Let us recall some definitions from the lecture.

Definition 1. *Fix a point $s \in \overline{C} \setminus C$, and let t be a local coordinate at s, defined in a neighborhood U of s. Denote $d := t\partial_t$ and let $\mathcal{D}'_C \subset \mathcal{D}_C$ be the sheaf of subalgebras generated by $\mathcal{O}_U$ and d. Let $j_s : C \cap U \hookrightarrow U$ be the embedding, and let $\mathcal{F} := (j_s)_*\mathcal{E}$. A $\mathcal{D}'_U$-submodule $\mathcal{E}' \subset \mathcal{F}$ is called a* lattice *if it is $\mathcal{O}_U$-coherent and $\mathcal{E}'|_{C\cap U} = \mathcal{E}$.*

We say that $\mathcal{E}$ is regular singular (RS) at s if there exist a neighborhood U and a lattice $\mathcal{E}' \subset \mathcal{F}$.

We say that $\mathcal{E}$ is regular singular (RS) if it is RS at all points $s \in \overline{C} \setminus C$. Denote the category of smooth regular singular $\mathcal{D}_C$-modules by $\mathcal{D}^{sm}_{RS}(C)$

(2) (P) Show that the category $\mathcal{D}^{sm}_{RS}(C)$ is closed under subquotients, extensions, duality, and $\otimes^!$.

(3) (P) Show that for any dominant map of smooth curves $\nu : C \to Y$, a smooth $\mathcal{D}_Y$-module $\mathcal{H}$ is RS if and only if $\nu^!\mathcal{H}$ is RS.

Definition 2. *A complex $\mathcal{F}$ of holonomic $\mathcal{D}_C$-modules is called RS if there exists an open dense subset $U \subset C$ such that all the cohomologies of $\mathcal{F}|_U$ are smooth and RS. Denote the category of such complexes (as a subcategory of the bounded derived category $D^b(\mathcal{D}_C)$ by $D^b_{RS}(\mathcal{D}_C)$.*

(4) (P) Let $\nu : C \to Y$ be a morphism of smooth (algebraic) curves. The category $D^b_{RS}(\mathcal{D}_C)$ of RS complexes is preserved by the functors $\nu^!$, ν_*, $\mathbb{D}$, and $\otimes^!$ (and thus also by $\nu_!$ and ν^*).

URL: http://www.wisdom.weizmann.ac.il/~dimagur/DmodII.html

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