EXERCISE 2 IN D-MODULES II

JOSEPH BERNSTEIN AND DMITRY GOUREVITCH

Lemma 1 (Noether Normalization Lemma). *Let k be an infinite field, and X be an algebraic variety over k isomorphic to the affine space A^n. Let $Y \subset X$ be a closed subvariety. If Y does not coincide with X then there exists a finite morphism $p : X \to A^n$ such that the image $p(Y)$ lies inside $A^{n-1} \subset A^n$*

(1) Take any proof of any formulation of Noether Normalization Lemma, and show that the proof implies the formulation given above.

(2) Let X be an affine algebraic variety of dimension m and $S \subset X$ a closed subvariety of dimension d.
   (i) Show that there exists a finite morphism $p : X \to A^M$ such that $p(S)$ lies inside A^d.
   (ii) More generally, suppose we have several closed subvarieties $S_1, ..., S_k$ of dimensions $d_1, ..., d_k$. Show that there exists a finite morphism $p : X \to A^m$ such that $p(S_i) \subset A^{d_i}$.

(3) (P) Let X be an irreducible algebraic variety of dimension n, $S \subset X$ is a closed subvariety of dimension $\leq n-2$. Let F be a quasicoherent O_X-module without torsion.
   Show that if the restriction of F to the subvariety $U = X \setminus S$ is coherent then F is coherent.
   Hint. Reduce to the case $X = A^n, S = A^{n-2}$. Realize F as a submodule of K^r, where K is the field of rational functions on X.

(4) (P) (P) Let X be an irreducible algebraic variety of dimension n, $S \subset X$ is a closed subvariety of dimension $\leq n-2$. Let F be a quasicoherent O_X-module without torsion.
   (i) Consider an an infinite increasing sequence of coherent O_X-modules $F_1 \subset F_2 \subset F_3 \subset ....$ Suppose that the sequence stabilizes on $X \setminus S$. Show that it stabilizes on X.
   (ii) Let $M \subset F$ be a coherent submodule. Let M' be the sum of all coherent submodules $N \subset F$ that include M' and satisfy $\dim supp(M/N) \leq \dim X - 2$. Then M' is coherent and itself satisfies $\dim supp(M'/N) \leq \dim X - 2$.

   In this situation, M' is called the *enhancement* of M.

   Algebraic RS modules.

   Let U be a smooth quasi-projective algebraic variety, and X be its good compactification. This means that X is projective and $S := X \setminus U$ is a divisor with strict normal crossings.

Definition 1. *Let $\mathcal{D}_{X,S} \subset \mathcal{D}_X$ denote the sheaf of subalgebras generated by $\mathcal{O}_X$ and by vector fields tangent to S. A coherent $\mathcal{D}_X$-module $\mathcal{F}$ is called* algebraic RS *(with respect to U) if its restriction $\mathcal{F}|_U$ is smooth, and $\mathcal{F}$ is a union of $\mathcal{O}_X$-coherent $\mathcal{D}_{X,S}$-submodules.*

(5) (P) The category of algebraic RS-modules (with respect to U) is closed w.r. to subquotients.

URL: http://www.wisdom.weizmann.ac.il/~dimagur/DmodII.html

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