EXERCISE 3 IN D-MODULES II

JOSEPH BERNSTEIN AND DMITRY GOUREVITCH

(1) Let X be an irreducible algebraic variety of dimension n and M a coherent O_X-module without torsion.

Definition 1. *We will call a* ***semi-small extension*** *of M a coherent O_X-module N without torsion that contains M, such that the support $supp(N/M)$ has dimension $\leq n-1$.*

We call this module a ***small extension*** *if $supp(N/M)$ has dimension $\leq n-2$.*

(i) Show that semi-small extensions $N \supset M$ are naturally realized as coherent O_X-submodules of the module $M_K = K \otimes_{O_X} M$, where K denotes the field of rational functions. In particular they form a partially ordered set.

(ii) Let us set $M^* := Hom(M, O_X)$. Show that this is a coherent O_X-module.

Construct a canonical morphism $i : M \to M^{**}$ and show that M does not have torsion if and only if $i : M \to M^{**}$ is an embedding.

(2) (P) Let X be isomorphic to the affine space $\mathbb{A}^n$.

(i) Let M be a free O_X-module. Show that M has no proper small extensions.

Hint. Reduce to the case when $X = \mathbb{A}^n$ and $S = supp(N/M) \subset \mathbb{A}^{n-2}$

(ii) Show that for a semi-small extension $N \supset M$ the natural morphism $N^* \to M^*$ is injective.

Show that for a small extension this morphism is an isomorphism.

(3) (P) Let X be an irreducible algebraic variety. Show that any coherent O_X-module M without torsion has a maximal small extension, i.e. there exists a small extension $N \supset M$ that contains all other small extensions.

Hint. Reduce to the case when $X \approx \mathbb{A}^n$.

Show that in this case the coherent O_X-module M^{**} contains all small extensions of M, and itself is a small extension.

URL: http://www.wisdom.weizmann.ac.il/~dimagur/DmodII.html

Date: May 10, 2021.