# EXERCISE 4 IN D-MODULES II

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**Local analytic connections.**

Consider an open unit disc $D \subset \mathbb{C}$ and denote by $D^*$ the punctured disc $D^* = D \setminus 0$. Let $E$ be a holomorphic vector bundle on $D$. We can trivialize it by choosing basis of sections.

Let $\nabla$ be a holomorphic connection on $E$ restricted to $D^*$. We say that $\nabla$ is meromorphic if for any holomorphic section $u$ of $E$ the section $\nabla(u)$ is a meromorphic section (it can have a pole only at 0). We say that $\nabla$ is regular at 0 if all sections $\nabla(u)$ are holomorphic at 0. We say that $\nabla$ is Fuchsian if all sections $\nabla(u)$ have poles of order $\leq 1$.

If we trivialize the vector bundle $E$ we can write the connection $\nabla$ as $\nabla(u) = \partial(u) + B(u)$, where $B$ is a meromorphic matrix valued function $B = B(t)$ (here $t$ is the parameter on $D$ and $\partial = \partial_t$).

The connection $\nabla$ is regular if the function $B(t)$ is regular at 0, and is Fuchsian if the function $B(t)$ has pole of order $\leq 1$ at 0.

(1) (i) Suppose the connection $\nabla$ is Fuchsian. Define the expansion matrix $A$ of $\nabla$ $A := tb(t)|_{t=0}$. Eigenvalues $\lambda_i$ of this matrix are called expansion coefficients of the connection $\nabla$.

(ii) Show that up to conjugation the expansion matrix $A$ (and hence expansion coefficients) do not depend on trivialization of the bundle $E$.

(2) Define the monodromy operator $\mu = \mu_\nabla$ of the connection $\nabla$ on $D^*$. Show that the matrix $\mu$ is conjugate to the matrix $\exp(2\pi i A)$.

(3) (P) Let $E, E'$ be holomorphic bundles with Fuchsian connections. Show that then the bundles $E \otimes E'$, $E^*$, $Hom(E, E')$ also have natural connections and these connections are Fuchsian.

Compute the expansion coefficients of these connections.

(4) (P) Let $E$ be a vector bundle with a Fuchsian connection, u a flat holomorphic section of this bundle on $D^*$ (such sections do not always exist).

(i) Show that the section $u$ extends to a meromorphic section on the whole disc $D$.

(ii) Suppose that every expansion coefficient $\lambda$ of the connection has $Re(\lambda) \geq -1$. Show that the section $u$ extends to a holomorphic section on the whole disc $D$.

(5) (P) Let $E, E'$ be two vector bundles on $D$ with Fuchsian connections on $D^*$.

(i) Show that a morphism : $E \to E'$ of vector bundles compatible with connections is the same as flat section $u$ of the vector bundle $Hom_O(E, E')$

(ii) Suppose we know that all expansion coefficients of $E$ and $E'$ have real parts $0 \leq \lambda < 1$. Show that any isomorphism of bundles $E, E'$ on the punctured disc $D^*$ compatible with connections holomorphically extends to the whole disc $D$.

(6) (P) Suppose we have a holomorphic vector bundle with connection $E^0$ on the punctured disc $D^*$.

(i) Show that there exists a vector bundle $E$ with Fuchsian connection on $D$ and an isomorphism $\alpha$ of the restriction of $E$ to $D^*$ with $E^0$.

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(ii) Show that if we assume that all expansion coefficients of the bundle $E$ are normalized (i.e. have real parts $\geq 0$ and $< 1$), then this pair $(E, \alpha)$ is defined uniquely up to a unique isomorphism.

(7) (*) Consider an analytic variety $X = D^n$. Consider a punctured variety $X^*$ obtained from $X$ by removing $k$ coordinate hyperplanes. Let $E^0$ be a holomorphic vector bundle on $X^*$ with flat connection. Formulate and prove statements analogous to problem 6 in this case.

*URL*: http://www.wisdom.weizmann.ac.il/~dimagur/DmodII.html