THE ALGEBRAIC THEORY OF D-MODULES

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Abstract. These are draft notes of the course I gave in fall semester 2015 at the Weizmann Institute of Science, under the guidance of Joseph Bernstein. Notes by Alexander Shamov. Typos are probable, mistakes possible, comments welcome.

Contents

1. D-modules on affine spaces

Notation 1.1. $\mathcal{D}_n := \langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$ is the algebra of polynomial differential operators.

Definition 1.2. Let M be a finitely generated $\mathcal{D}_n$-module. Then a solution of M is a homomorphism from M to some $\mathcal{D}_n$-module of functions (say, $C^\infty(\mathbb{R}^n)$ or $C^{-\infty}(\mathbb{R}^n)$).

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Example 1.3. To any linear system of PDEs $\{L_1 f = 0, \ldots, L_k f = 0\}$, we associate the $\mathcal{D}_n$-module $\mathcal{D}_n / \langle L_1, \ldots, L_k \rangle$.

Fix an algebraically closed field $\mathbb{K}$ of characteristic 0.

Definition 1.4. $\mathcal{D}_n$ is the subalgebra of $\operatorname{End}_{\mathbb{K}} \mathbb{K}[x_1, \ldots, x_n]$ generated by derivations and multiplication operators.

Exercise 1.5.

$$\mathcal{D}_n = \langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle / \langle [x_i, x_j] = 0, [\partial_i, \partial_j] = 0, [\partial_i, x_j] = \delta_{ij} \rangle .$$

Exercise 1.6. The center is: $Z(\mathcal{D}_n) = \mathbb{K}$.

Notation 1.7. $\mathcal{M}$ is the category of left $\mathcal{D}_n$-modules, $\mathcal{M}^r$ is that of right modules, $\mathcal{M}^f$ — finitely generated left modules.

1.1. **Dimension.**

Lemma 1.8. *For any $M \in \mathcal{M}(D_n)$ either $M = 0$ or $\dim_{\mathbb{K}} M = \infty$.*

Proof. $0 = \operatorname{tr}[x_1, \partial_1] = \operatorname{tr} 1 = \dim_{\mathbb{K}} M$. ☐

This motivates other ways of measuring the “size” of a module.

Definition 1.9. A filtered algebra is an algebra A equipped with an increasing sequence of subspaces $F^i A, i \geq 0$, $F^i A \subset F^{i+1} A$, $\bigcup_i F^i A = A$, such that $1 \in F^0 A$ and $F^i A \cdot F^j A \subset F^{i+j} A$. A filtration is called good if $F^i A$ is f.g. over $F^0 A$, and $F^{i+1} A = F^1 A \cdot F^i A$ for $i \gg 0$ (i large enough).

Example 1.10. $A = \mathbb{K}[y_1, \ldots, y_m]$, $F^i A := \{\deg \leq i\}$.

Example 1.11. Bernstein filtration: $F^i \mathcal{D}_n := \operatorname{span}\left\{ x^\alpha \partial^\beta \,\middle|\, |\alpha| + |\beta| \leq i \right\}$

Definition 1.12. For a filtered algebra $(F^i A)$ the associated graded algebra is $\operatorname{Gr}_F A := \bigoplus_i (F^i A / F^{i-1} A)$, $F^{-1} A := 0$.

Example 1.13. $\operatorname{Gr} \mathcal{D}_n = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_n]$.

Let A be a good filtered algebra.

Definition 1.14. A filtered A-module M is a module equipped with an increasing sequence of subspaces $F^i M$, such that $F^i A \cdot F^j M \subset F^{i+j} M$, $\bigcup_i F^i M = M$. A filtration is called good if $F^i M$ are finitely generated over $F^0 A$, and $F^{i+1} M = F^1 A \cdot F^i M$ for $i \gg 0$.

Exercise 1.15. Any two good filtrations are comparable, i.e. $F^{i-m} M \subset \Phi^i M \subset F^{i+m} M$.

Proof. Suppose that $\forall i : F^{N+i} M = (F^1 A)^i F^N M$. Since $F^N M$ is finitely generated over $F^0 A$, and $\bigcup_i \Phi^i M = M$, one can assume that $F^N M \subset \Phi^{N'} M$. WLOG, $\Phi^{N'+i} M = (F^1 A)^i \Phi^{N'} M$. By the same argument, $\Phi^{N'} M \subset F^{N''} M$. Now $F^{N+i} M \subset \Phi^{N'+i} M \subset F^{N''+i} M$ for all $i \geq 0$. ☐

Remark 1.16. A filtered algebra is good iff it's good as a module over itself.

We define $\operatorname{Gr}_F M$ for a filtered module in a similar way as for algebras. We will sometimes write A^i for $F^i A$ and $M^i$for $F^i M$ if the filtration is understood.

Exercise 1.17. Assume that A^i is a good filtered algebra. Then F^iM is good iff $\mathrm{Gr}_F M$ is finitely generated over $\mathrm{Gr}_F A$.

Proof. Suppose that $M^i := F^iM$ is good, and $M^{i+1} = A^1M^i$ starting from some N. Take generators m_i of M^N over A^0. Then they their symbols generate $\mathrm{Gr}\, M$ over $\mathrm{Gr}\, A$. Conversely, suppose that $\mathrm{Gr}\, M$ is finitely generated over $\mathrm{Gr}\, A$ by elements m_i, $\deg m_i = d_i$. Then the filtras M^i are obtained by iterated extensions from M^j/M^{j-1}, which are finitely generated over A^0, so M^i are also finitely generated over A^0. On the other hand,

$$A^1M^i/M^i \simeq \left(A^1/A^0\right)\left(M^i/M^{i-1}\right)$$

and if A is good then

$$M^{i+1}/M^i = \sum_j \left(A^{i-j+1}/A^{i-j}\right)\left(M^j/M^{j-1}\right) =$$

$$\left(A^1/A^0\right)\sum_j \left(A^{i-j}/A^{i-j-1}\right)\left(M^j/M^{j-1}\right) = \left(A^1/A^0\right)\left(M^i/M^{i-1}\right)$$

where j runs through the degrees of the generators of $\mathrm{Gr}\, M$ over $\mathrm{Gr}\, A$ and i is assumed to be larger than the maximum of these degrees. ☐

Exercise 1.18. Fix a good filtered algebra. M admits a good filtration iff it's finitely generated.

Proof. Suppose first that M admits a good filtration. Then $M^{i+1} = A^1M^i$ for $i \geq N$ and $M^N$is finitely generated over A^0. Thus the generators of M^Nover A^0 generate M. Conversely, assume M is generated by a finite set $x_{i\,i=1}^{\;k}$ and consider the filtration $M^i := A^ix_1 + \cdots + A^ix_k$. Since A^i is finitely generated over A^0, M^i is also finitely generated over A^0, and $M^{i+1} = A^1M^i$ as long as $A^{i+1} = A^1A^i$. Since any element $m \in M$ is representable as $\sum_{i=1}^k a_ix_i$, we have $m \in M^n$, where n is big enough so that $a_i \in A^n$.Therefore $M = \bigcup_i M^i$. ☐

Definition 1.19. For a filtered module F^iM and a short exact sequence $0 \to L \to M \to N \to 0$, define induced filtrations on L and M by $F^iL := F^iM \cap L$ and $F^iN := F^iM/F^iL$. A map $f : A \to B$ of filtered modules is called strict if $f\left(A^i\right) = f\left(A\right) \cap B^i$.

Exercise 1.20. Let $L \to M \to N$ be an exact sequence of filtered modules and strict maps between them. Then the corresponding sequence $\mathrm{Gr}\, L \to \mathrm{Gr}\, M \to \mathrm{Gr}\, N$ is also exact.

Exercise 1.21. For a good filtered module F^iM and a short exact sequence $0 \to L \to M \to N \to 0$, the induced filtration on N is good, and if $\mathrm{Gr}\, A$ is Noetherian then so is the induced filtration on L.

Proof. Note that all maps in $0 \to L \to M \to N \to 0$ are strict. Thus $\mathrm{Gr}\, N$ is a factor of $\mathrm{Gr}\, M$, so it's finitely generated over $\mathrm{Gr}\, A$; and $\mathrm{Gr}\, L$ is a submodule of $\mathrm{Gr}\, M$, so if $\mathrm{Gr}\, A$ is Noetherian then $\mathrm{Gr}\, L$ is finitely generated over it. ☐

Remark 1.22. The category of filtered modules is not Abelian — say, the shift map doesn't have a cokernel.

Theorem 1.23. *Suppose that A is good and $\mathrm{Gr}_F A$ is Noetherian. Then so is A.*

Proof. Let M be a finitely generated A-module, and $L \subset M$. Pick a good filtration F. Then $\operatorname{Gr}_F M$ is finitely generated, $\operatorname{Gr}_F A$ is Noetherian, hence $\operatorname{Gr}_F L$ is finitely generated, so $F^i L$ is good, so L is finitely generated over A. ☐

Corollary 1.24. *$\mathcal{D}_n$ is Noetherian. The universal enveloping algebra of any finite dimensional Lie algebra is Noetherian.*

Notation 1.25. If $f, g : \mathbb{N} \to \mathbb{Z}$ we say $f \sim g$ if $f = g$ for $i \gg 0$. $\Delta f(i) := f(i+1) - f(i)$.

Fact 1.26. *Integer-valued polynomials have the form $f(i) = \sum_{k=0}^{d} c_k \binom{i}{k}$, $c_k \in \mathbb{Z}$.*

Theorem 1.27 (Hilbert)**.** *Let $R = \bigoplus R^i$ be a graded finitely generated $\mathbb{K}[x_1, \ldots, x_n]$-module. Then $b(i) := \dim_{\mathbb{K}} R^i$ is eventually polynomial of degree $\leq n$, called the functional dimension of R.*

Proof. Define $(R[1])^i := R^{i+1}$.

$$0 \to \ker x_n \to R \stackrel{x_n}{\to} R[1] \to \operatorname{coker} x_n \to 0$$

This is a morphism of graded modules, so $\ker x_n$ and $\operatorname{coker} x_n$ are graded modules. Thus:

$$\dim_{\mathbb{K}} (\ker x_n)^i - \dim_{\mathbb{K}} R^i + \dim_{\mathbb{K}} R^{i+1} - \dim_{\mathbb{K}} (\operatorname{coker} x_n)^i = 0$$

On the other hand, x_n acts by 0 on both ker and coker, so by induction on n we know that $\Delta \dim_{\mathbb{K}} R^i$ is eventually polynomial, therefore so is $\dim_{\mathbb{K}} R^i$. ☐

Corollary 1.28. *Let $F^i M$ be a good filtered $\mathcal{D}_n$-module. Then $b_M(i) := \dim F^i M$ is eventually polynomial of degree $\leq 2n$.*

Proof. $b_M = b_{\operatorname{Gr} M}$, and note that $\operatorname{Gr} \mathcal{D}_n \simeq \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. ☐

Remark 1.29. Now since for any two filtrations $F^{i-k} M \subset \Phi^i M \subset F^{i+k} M$, the degree and leading coefficient are invariant (for a fixed filtration of the algebra).

Definition 1.30. The degree of $b_M(i)$ is called the dimension of M and denoted $d(M)$, and the leading coefficient of $d(M)! b_M(i)$ is called the (Bernstein) degree of M and denoted $e(M)$.

Theorem 1.31 (Bernstein inequality)**.** *Let M be a finitely generated $\mathcal{D}_n$-module. If $M \neq 0$ then $n \leq d(M) \leq 2n$.*

Remark 1.32. Note that $d(M) \neq 0$ is equivalent to $\dim_{\mathbb{K}} M = \infty$, so Bernstein's inequality can be viewed as a generalization of that.

1.2. **Proof of Bernstein's inequality.** Define $N^0 := \ker x_n \subset M$, $N^\ell := \partial_n^\ell N^0$. They are viewed as $\mathcal{D}_{n-1}$-modules.

Lemma 1.33. *$\partial_n^\ell : N^0 \simeq N^\ell$ and N^ℓ are linearly independent.*

Proof. Let $m \in N^0$, $m_\ell := \partial_n^\ell m$. Then since $\left[x_n, \partial_n^\ell\right] = -\ell \partial_n^{\ell-1}$, we have

$$x_n m_\ell = -\ell m_{\ell-1}$$

Thus x_n "inverts" ∂_n on N^ℓ (like in reps of $\mathfrak{sl}_2$). Thus $\partial_n^\ell : N^0 \simeq N^\ell$.

N^ℓ are linearly independent because they are different eigenspaces of $x_n \partial_n$. ☐

Corollary 1.34. *If $\ker x_n \neq 0$ and Bernstein's inequality holds for $\mathcal{D}_{n-1}$ then $d(M) \geq n$.*

We use notation $\mathcal{D}_n^\ell := F^\ell \mathcal{D}_n$.

Proof. Let $m \neq 0$, $m \in \ker x_n$. Then $\mathcal{D}_n^{2\ell} m \supset \bigoplus_{i=0}^{\ell} \partial_n^i \mathcal{D}_{n-1}^\ell m = \bigoplus_{i=0}^{\ell} \mathcal{D}_{n-1}^\ell \partial_n^i m$. Thus $\dim \mathcal{D}_n^{2\ell} m \geq \ell \dim \mathcal{D}_{n-1}^\ell m \geq \mathrm{const} \cdot \ell^n$. ☐

Corollary 1.35. *If* $\operatorname{coker} x_n \neq 0$ *and Bernstein's inequality holds for* $\mathcal{D}_{n-1}$ *then* $d(M) \geq n$.

Proof. By the previous corollary we can assume $\ker x_n = 0$. Now, $\operatorname{coker} x_n = M/x_n M$ is a $\mathcal{D}_{n-1}$-module. Assume it's finitely generated. Then $\dim F^i M - \dim x_n F^{i-1} M \geq ci^{n-1}$. If it's not finitely generated then take a finitely generated submodule. Thus $\Delta \dim F^i M \geq ci^{n-1}$, so $d(M) \geq n$. ☐

Exercise 1.36 (Amitsur-Kaplansky lemma)**.** Let $\mathbb{L}$ be an uncountable algebraically closed field. Let V be an $\mathbb{L}$-vector space of countable dimension. Then any linear operator on $T : V \to V$ has nonempty spectrum, i.e. $T - \lambda is$not invertible for some $\lambda \in \mathbb{L}$.

Proof. Assume by way of contradiction that the spectrum of T is empty. Let $v \in V$ be a non-zero vector. Then $(T-\lambda)^{-1} v$, $\lambda \in \mathbb{L}$ is an uncountable set. Thus it is linearly dependant. Picking a dependence and bringing it to a common denominator we obtain $p(T)v = 0$, for some polynomial p. On the other hand, p is a product of linear factors, thus $p(T)$ is invertible and has no kernel. Contradiction. ☐

Proof of Bernstein inequality. Extend the field so that it becomes uncountable. By the previous lemma, $x_n - \lambda$ is not invertible for some λ. Now apply the automorphism $x_n \mapsto x_n - \lambda$. Now the theorem follows by induction from the previous corollaries. ☐

2. Holonomic modules, and an application.

2.1. Another proof of Bernstein inequality.

Lemma 2.1. *(Exc). The center of* $\mathcal{D}_n$ *is* $\mathbb{K}$.

Lemma 2.2. *Let* $F^i M$ *be a good filtration on a* $\mathcal{D}_n$*-module* M*. Then the action defines an embedding of* $\mathcal{D}_n^i$ *into* $Hom_k(F^i M, F^{2i} M)$.

Proof. The map is defined by definition of filtration. Let us prove that it is an embedding by induction on i. For $i = 0$ this is obvious. For a bigger i, let $d \neq 0$ lie in the kernel. Since d is not scalar and thus not central, there exists l such that $[d, x_l] \neq 0$ or $[d, \partial_l] \neq 0$. Assume WLOG $[d, x_1] \neq 0$. Then $[d, x_1] \in \mathcal{D}_n^{i-1}$and by the induction hypothesis $[d, x_1]v \neq 0$ for some $v \in F^{i-1}M$. However, $[d, x_1]v = dx_1 v - x_1 dv = 0$, since $v, x_1 v \in F^i M$. We arrived at a contradiction and thus $d = 0$. ☐

Joseph's Proof of Bernstein inequality. Suppose by way of contradiction that $d(M) \leq n-1$. Then $\dim Hom_k(F^i M, F^{2i} M) < ci^{n-1}(2i)^{n-1} = c' i^{2n-2}$. On the other hand $\dim \mathcal{D}_n^i > c'' i^{2n}$. This contradicts the previous lemma. ☐

In the nest section we will state without proof a deep geometric theorem that implies the Bernstein inequality.

Definition 2.3. A finitely generated $\mathcal{D}_n$-module M is called **holonomic** if $d(M) = n$.

Exercise 2.4. If M is holonomic then it has length at most $e(M)$.

Corollary 2.5. *(Exc). Let M be a $\mathcal{D}_n$-module, and let F^iM be a (not necessary good) filtration on M. Suppose that* $\dim F^iM \leq e\binom{i}{n}$ *for some e. Then it is finitely-generated. Moreover, it is holonomic and of length at most e.*

We are now ready to give the first application to the theory of distributions. Let P be a polynomial in n real variables. Let $\lambda \in \mathbb{C}$with $Re\lambda > -1$ and consider the locally integrable function $|P|^\lambda$.

Theorem 2.6. *(Bernstein, Gelfand, Gelfand, Atiya, ...) Consider $|P|^\lambda$ as a family of generalized functions. Then this family has meromorphic continuation to the entire complex plane with poles in a finite number of arithmetic progressions.*

This theorem follows from an algebraic statement saying that there exists a differential operator d with polynomial coefficients (that depend also on λ), and a polynomial b in λsuch that $d|P|^\lambda = b|P|^{\lambda-1}$. Let us formulate this algebraic statement more precisely, over any field, and prove it.

Notation 2.7. Fix a polynomial $P \in k[x_1, \ldots, x_n]$. Let $K := k(\lambda)$ be the field of rational functions. Consider the $\mathcal{D}_n(K)$-module $M_P := M'_p \otimes_{k[\lambda]} K$, where $M'_p := spanQP^{\lambda-l}$, where $Q \in k[x_{1,} \ldots, x_{n,}\lambda]$ and $l \in \mathbb{Z}_{\geq 0}$, with the relations $PP^{\lambda-l} = P^{\lambda-l+1}$, and the action of $\mathcal{D}_n[\lambda]$ given by

$$\partial_i(QP^{\lambda-l}) = \partial_i(Q)P^{\lambda-l} + Q(\lambda - l)\partial_i(P)P^{\lambda-l-1}.$$

Lemma 2.8. *The module $M_P$is finitely generated, and, moreover, holonomic.*

Proof. Define a filtration on M_P by

$$F^iM_P := QP^{\lambda-i}s.t. \deg Q \leq (\deg P + 1)i.$$

It satisfies $\dim F^iM_P \leq ci^n$. It's not clear whether this is a good filtration, but by the Corollary above we still get that $M_P$is finitely generated and holonomic. ☐

Corollary 2.9. *There exist $d \in k[x_1, \ldots, x_n, \partial_{1,} \ldots, \partial_{n,}\lambda]$ and $b \in k[\lambda]$ s.t. $dP^\lambda = bP^{\lambda-1}$.*

Proof. Consider the increasing chain of submodules

$$\mathcal{D}_n(K)P^\lambda \subset \mathcal{D}_n(K)P^{\lambda-1} \subset \ldots.$$

This chain has to stabilize. Thus $\tilde{d}P^{\lambda-k} = P^{\lambda-k-1}$ for some $\tilde{d} \in \mathcal{D}_n(K)$. Applying the automorphism $\lambda \mapsto \lambda + 1$we get that $\hat{d}P^\lambda = P^{\lambda-1}$ for some $\hat{d} \in \mathcal{D}_n(K)$. Now, we can decompose $\hat{d} = \frac{d}{b}$. ☐

Finally, let us show that holonomic modules are cyclic.

Theorem 2.10. *Let R be a simple Noetherian non-Artinian ring, and M a finitely generated Artinian left R-module. Then M is cyclic.*

Proof. By induction on length, we assume that $M = R\langle u, v\rangle$ with Rv simple. Since R is not Artinian, there is d, such that $du = 0$. On the other hand, since R is simple, $R = RdR$, so there is b, such that $dbv \neq 0$. Now $M = R\langle u + bv\rangle$. $d(u + bv) = dbv \subset Rv$, so since Rv is simple, $\langle u + bv\rangle \supset Rv$. Thus $bv \in \langle u + bv\rangle$, so $u \in \langle u + bv\rangle$. ☐

Corollary 2.11. *Holonomic $\mathcal{D}_n$-modules are cyclic.*

3. Associated varieties and singular support

Let A be a finitely-generated commutative $\mathbb{K}$-algebra without nilpotents.

Definition 3.1. Let M be an A-module. Denote by $\operatorname{Ann} M$ the annihilator ideal $AnnM := \{a \in A \,|aM = 0\}$ and define the support $\operatorname{Supp}M$ to be the zeros of $AnnM$ in the maximal spectrum $\operatorname{Specm} A$.

If M is finitely generated then $\operatorname{Supp}M$ is the support of the coherent sheaf on $\operatorname{Spec} A$ that corresponds to M. This follows from Nakayama's lemma. If $A = \mathbb{K}[x_1, \dots, x_n]$ then $\operatorname{Specm} A = \mathbb{A}^n$.

The algebra $\mathcal{D}_n$ is not commutative, and in order to associate a variety to a finitely-generated $\mathcal{D}_n$ module we will use the associated graded algebra $\mathbb{K}[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$

Definition 3.2. Two modules M, N over the same algebra are called Jordan-Holder equivalent if there exist two increasing chains of the same finite length of submodules $0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$ and $0 = N_0 \subset N_1 \subset \cdots \subset N_m = N$ and a permutation $\sigma \in Sym_m$ s.t. $M_i/M_{i-1} \simeq N_{\sigma(i)}/N_{\sigma(i)-1}$ for any i.

Lemma 3.3. *Let F, Φ be two good filtrations on a $\mathcal{D}_n$-module M. Then $Gr_F M$ and $Gr_\Phi M$ are Jordan-Holder equivalent.*

Proof. Case 1. F, Φ are neighbors, i.e. $F^i M \subset \Phi^i M \subset F^{i+1} M \subset \Phi^{i+1} M$. In this case we have a well-defined map $\phi : Gr_F M \to Gr_\Phi M$, and $Ker\phi \simeq CoKer\phi$. Thus $Gr_F M$ and $Gr_\Phi M$ are Jordan-Holder equivalent.

In the general case, one can construct a sequence of neighboring filtrations $F^i M + \Phi^{i+l} M$, which starts with F and ends with a shift of Φ. $\square$

Lemma 3.4. *Let $0 \to L \to M \to N \to 0$ be a short exact sequence of A-modules. Then* $\operatorname{Supp}M = \operatorname{Supp}N \cup \operatorname{Supp}L$.

Proof. Clearly $\operatorname{Ann} M \subset \operatorname{Ann} N \cap \operatorname{Ann} L$. Now, if $a \in \operatorname{Ann} N \cap \operatorname{Ann} L$ then for any $m \in M$ we have $am \in L$ and thus $a^2 m = 0$. This shows that $\operatorname{Ann} N \cap \operatorname{Ann} L \subset \operatorname{Rad} \operatorname{Ann} M$. So $\operatorname{Ann} M \subset \operatorname{Ann} N \cap \operatorname{Ann} L \subset \operatorname{Rad} \operatorname{Ann} M$ and thus their zero sets coincide. $\square$

Corollary 3.5. *If two A-modules are Jordan-Holder equivalent then they have the same support.*

Definition 3.6. The associated variety $AV(M)$ of a finitely-generated $\mathcal{D}_n$-module M is the support of $Gr_F M$ for some good filtration F.

By definition, $AV(M)$ is a closed subset of the affine space $\mathbb{A}^{2n}$. By Theorems 3.3 and 3.5 it does not depend on the choice of a good filtration.

Lemma 3.7 (Bernstein)**.** *Let M be a $\mathcal{D}_n$-module generated by a finite subset $S \subset M$. Let $I \in \mathcal{D}_n$ be the annihilator of S, and let $J \subseteq A := \mathbb{K}[x_1, ..., x_n, \xi_1, ..., \xi_n]$ be the ideal generated by the symbols of the elements of I. Show that the associated variety $AV(M)$ is the zero set of J.*

Proof. We first show that J vanishes on $AV(M)$. Let $S = \{m_1, \dots, m_s\}$. Define a good filtration on M by $F_i(M) = \mathcal{B}_i m_1 + ...\mathcal{B}_i m_s$, where $\mathcal{B}_i \subset \mathcal{D}_n$ is the $i-$th Bernstein filtration. If $d \in I \cap \mathcal{B}_j$ satisfies $dm_j = 0$ for any $m_i \in S$, then for any $c \in \mathcal{B}_i$ we have

$$dcm_j = [d, c]m_l + cdm_l = [d, c]m_l \in F_{i+j-1} M.$$

Thus, $\sigma(d)\widetilde{m_l} = 0$ where $\sigma : \mathcal{D}_n \longrightarrow A$ is the symbol map, and $\widetilde{m_l}$ is the image of m_l in $gr_F M$. Since $\{\widetilde{m_l}\}_{l=1}^s$ generate $\operatorname{Gr}_F M$, we get that $\sigma(d) \subset \operatorname{Ann}(\operatorname{Gr}_F(M)$. Thus $J \subset Ann(\operatorname{Gr}_F(M)$ and thus J vanishes on $AV(M)$.

Let us now show by induction on s that $Ann(\operatorname{Gr}_F(M) \subset Rad(J)$. It is enough to show that for any homogeneous polynomial $a \in Ann(\operatorname{Gr}_F(M)$, there exist a natural number t and an operator $d \in I$ such that $\sigma(d) = a^t$.

For $s = 1$ note that by definition of $\operatorname{Gr}_F M$, there exist operators $c, c' \in \mathcal{D}_n$ such that $c \in \mathcal{B}_{deg(a)}, c' \in \mathcal{B}_{deg(a)-1}$, $\sigma(c) = a$, and $cm_1 = c'm_1$. Then $d := c - c' \in I$ and $\sigma(d) = a$.

For the induction step, we will repeatedly use the for any submodule $L \subset M$, we have $AV(L) \subset AV(M)$. This is so since $\operatorname{Gr}_{F'} L \subset \operatorname{Gr}_F(M)$, where F' is the induced filtration on L. Note also that a vanishes on $AV(M)$.

Let $S_1 := \{m_1, \dots, m_{s-1}\}$, let $I_1 \subset \mathcal{D}_n$ denote its annihilator, and L_1 denote the submodule of M generated by S_1. Since $AV(L_1) \subset AV(M)$, a vanishes on $AV(L_1)$ and thus the induction hypothesis implies that there exist $d_1 \in I_1$ and a power t_1 such that $\sigma(d_1) = a^{t_1}$. Let $S_2 := \{d_1 m_s\}$ and L_2 be the submodule of M generated by it. Since $AV(L_2) \subset AV(M)$, a vanishes on $AV(L_2)$ and thus the base of the induction implies that there exist $d_2 \in \mathcal{D}_n$ and a power t_2 such that $d_2 d_1 m_s = 0$ and $\sigma(d_2) = a^{t_2}$. Now take $d := d_2 d_1$ and $t := t_1 + t_2$. □

Now we would like to argue that the dimension of $AV(M)$ equals $d(M)$. This follows from Hilbert's definition of dimension.

Definition 3.8. Let $X \subset \mathbb{A}^n$ be an affine algebraic variety and let $I \subset A := \mathbb{K}[x_1, \dots, x_n]$ be the ideal of functions vanishing on I. The standard filtration on A induces a good filtration F^i on A/I. By Theorem 1.27, the function $f(i) := \dim F^i(A/I)$ is eventually polynomial. Define $\dim X$ to be the degree of this polynomial.

Exercise 3.9. For any $M \in \mathcal{M}^f(\mathcal{D}_n)$, $\dim AV(M) = d(M)$.

3.1. **Digression on several definitions of dimension of algebraic varieties.** Let us first define dimension by properties and then discuss several equivalent definitions.

Definition 3.10. A dimension is a correspondence of a non-negative integer to every algebraic variety such that

(i) $\dim(\mathbb{A}^n) = n$
(ii) For a (locally closed) subvariety $Y \subset X$, $\dim(X) = \max(\dim Y, \dim(X \setminus Y))$.
(iii) For a finite epimorphism $\varphi : X \to Y$, $\dim X = \dim Y$.

The uniqueness of dimension follows from the Noether normalization lemma.

Lemma 3.11. *For any affine algebraic variety X, there exists a finite epimorphism $\varphi : X \to \mathbb{A}^n$ for some n.*

A finite morphism is a morphism $\varphi : X \to Y$ such that for any open affine $U \subset Y$, the preimage $\varphi^{-1}(U)$ is affine and the algebra $\mathcal{O}(\varphi^{-1}(U))$ of regular functions on it is finitely-generated as a module over $\mathcal{O}(U)$. Finite morphisms are proper and have finite fibers.

There are several constructions of the dimension function. One of them is the Krull dimension: the maximal length of a strictly increasing chain of closed irreducible non-empty subsets, minus one. Another is the Hilbert dimension: define the dimension of a variety as the maximal among the dimension of open affine subvarieties, and for an affine subvariety use Definition 3.8. Another way is to define the dimension of an affine variety to be the transcendence degree of its field of rational functions (over $\mathbb{K}$).

3.2. **The geometric filtration.** There is another very natural filtration on the algebra $\mathcal{D}_n$ - filtration by the degree of the differential operator. In other words, $\deg x_i = 0$, $\deg \partial_i = 1$. This filtration is called *the geometric filtration.*

Note that the associated graded algebra by this filtration is again isomorphic to $\mathbb{K}[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$, but with a different grading. Note also that this is a good algebra filtration, and all the lemmas we proved about the arithmetic filtration hold for the geometric filtration, with one exception: the geometric filtras are infinite dimensional. Thus we cannot define a "geometric dimension", but we can define a "geometric associated variety". It is called the singular support, or the characteristic variety.

Definition 3.12. Let $M \in \mathcal{M}^f(\mathcal{D}_n)$ and let F be a filtration on M which is good with respect to the geometric filtration on $\mathcal{D}_n$. Define the singular support of M to be $\mathrm{SingSupp}(M) := \mathrm{Supp}(\mathrm{Gr}_F M)$.

Proposition 3.13. *$d(M) = \dim \mathrm{SingSupp}(M)$.*

We will now sketch an elementary proof, and give a deeper proof in the next section.

Sketch of proof. It is enough to prove the proposition for a cyclic module $M = \mathcal{D}_n/I$. Consider a sequence of filtrations F_l^i on $\mathcal{D}_n$ given by $\deg_l(x_i) = 1$, $\deg_l(\partial_i) = l$. Then for any $d \in I$ and for l big enough, the symbol of d with respect to F_l is the highest homogeneous summand of the symbol of d with respect to the Bernstein filtration. Thus it is enough to show that $\dim \mathrm{Supp}\,\mathrm{Gr}_{F_l} M = \dim \mathrm{Supp}\,\mathrm{Gr}_{F_{l+1}} M$ for every l, where $F_i^l M = F_i^l(\mathcal{D}_n)/(I \cap F_i^l(\mathcal{D}_n))$. By Hilbert's definition of dimension this amounts to computing that the eventual-polynomial functions $\dim F_i^l M$ and $F_i^{l+1} M$ have the same degrees.

Warning: the filtrations F_l on $\mathcal{D}_n$ are not good by our definition. However, they are still "almost" good, namely the Rees algebra $\bigoplus_{i\in\mathbb{Z}} t^i F_l^i \mathcal{D}_n$ is finitely generated. It is possible to work with such filtrations in a similar way to good filtrations. ☐

3.3. **Involutivity of the associated variety.** The affine space $\mathbb{A}^{2n}$ has a natural symplectic form. On the tangent space at zero it is given by

$$\omega(x_i, x_j) = \omega(y_i, y_j) = 0, \omega(x_i, y_j) = \delta_{ij}.$$

Extending this formula by Leibnitz rule we get the Poisson brackets on the whole algebra $k[x_1, \dots, x_n, y_1, \dots, y_n]$. In fact, these Poisson brackets can be obtained from $\mathcal{D}_n$: for any two homogeneous polynomials $a, b \in k[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$ choose differential operators $c, d \in \mathcal{D}_n$ with symbols a, b. Then, a, b is the symbol of $[a, b]$. Another way to obtain this form is to identify $\mathbb{A}^{2n}$ with the cotangent bundle $T^*\mathbb{A}^n$.

Definition 3.14. An algebraic subvariety X of $\mathbb{A}^{2n}$ is called *coisotropic* or *involutive* or *integrable* if the ideal of polynomials that vanish on X is stable under the Poisson brackets.

Remark 3.15. This is equivalent to saying that the tangent space to X at every smooth point includes its orthogonal complement inside the tangent space to $\mathbb{A}^{2n}$ w.r. to the symplectic form.

Theorem 3.16 (Gabber, Kashiwara-Kawai-Sato)**.** *For any $M \in \mathcal{M}^f(\mathcal{D}_n)$, both $AV(M)$ and $\mathrm{SingSupp} M$ are coisotropic.*

Note that any coisotrpoic subvariety has dimension at least n, and thus this theorem implies the Bernstein inequality.

The proof of this theorem is outside the scope of our course. It is not difficult in fact to show that $AnnGr(M)$ is closed under the Poisson brackets. The difficulty is to show that so does its radical. We will not use the theorem and the corollary, this was just to give a geometric intuition. This theorem has applications to the theory of invariant distributions, in addition to the ones that Bernstein's inequality does.

Since $\mathrm{SingSupp} M$ is (almost by definition) invariant under homotheties in $\xi_1, \ldots, \xi_n$, Theorem 3.16 implies the following corollary.

Corollary 3.17. *For any holonomic $M \in \mathcal{M}^f(\mathcal{D}_n)$, $\mathrm{SingSupp} M$ is a finite union of conormal bundles to closed subvarieties of $\mathbb{A}^{2n}$.*

3.4. **Irreducible non-holonomic $\mathcal{D}_n$-modules.** We will now show that there are many irreducible non-holonomic $\mathcal{D}_n$-modules.

Definition 3.18. We call a coisotropic homogeneous closed subvariety of $\mathbb{A}^{2n}$ minimal if it's minimal among such.

Theorem 3.19. *Let $d \in \mathcal{D}_n$, such that $\sigma(d)$ is irreducible, and $Z(\sigma(d))$ is coisotropic and minimal. Then the left ideal $\mathcal{D}_n d$ is maximal, so that $\mathcal{D}_n/\mathcal{D}_n d$ is irreducible of dimension $2n-1$ over $\mathcal{D}_n$.*

Proof.

$$0 \to \mathcal{D}_n d \to \mathcal{D}_n \to \mathcal{D}_n/\mathcal{D}_n d \to 0$$

$$0 \to \mathrm{Gr}\,\mathcal{D}_n d \to \mathbb{K}[x_1, \ldots, x_{2n}] \to \mathrm{Gr}(\mathcal{D}_n/\mathcal{D}_n d) \to 0$$

$$\mathrm{Ann}\,\mathrm{Gr}(\mathcal{D}_n/\mathcal{D}_n d) \simeq \mathbb{K}[x_1, \ldots, x_n]\,\sigma(d)$$

Assume that $\mathcal{D}_n d \subset J$ for some $J \neq \mathcal{D}_n$. Then

$$0 \to J/\mathcal{D}_n d \to \mathcal{D}_n/\mathcal{D}_n d \to \mathcal{D}_n/J \to 0$$

$$Z(\sigma(J)) \subsetneq Z(\sigma(d))$$

because otherwise $\mathrm{rad}\,\langle\sigma(J)\rangle = \mathrm{rad}\,\langle\sigma(d)\rangle = \langle\sigma(d)\rangle$, in which case $J = \langle d\rangle$. But then by minimality of $Z(\sigma(d))$, we should have $Z(\sigma(J)) = \varnothing$, so that $J = \mathcal{D}_n$. ☐

Theorem 3.20 (Bernstein-Luntz). *The property $\{Z(f)$ is minimal$\}$ holds generically.*

4. Operations on D-modules

1. Fourier transform maps Schwartz functions into Schwartz measures and vice versa. It also maps tempered generalized functions to tempered distributions. It also maps product into convolution and $\widehat{x_j f} = (i/2\pi)\partial_j \hat{f}$, $\widehat{\partial_j f} = 2\pi i x_j \hat{f}$.

The corresponding operation on $\mathcal{D}_n$-modules is just switching the actions of x_j and ∂_j. Let us give an application to PDE. Let d be a differential operator on $\mathbb{R}^n$ with constant real coefficients, and h be a smooth function. We are looking for a solution of the equation $df = h$. First of all, it is enough to find a solution for $d\xi = \delta_0$ in distributions, because then the convolution $\xi * f$ will solve the original equation. Now, applying Fourier transform we get the equation $pg = 1$, where p is the polynomial obtained from d by replacing all ∂_j by $2\pi i x_j$, and g is the unknown generalized function. Then it is clear for us that g should be p^{-1}. This is not well-defined a-priory, since p might have zeros. However, $(p^2)^\lambda$, as we have shown, is defined as a meromorphic distribution-valued function in λ. It might have a pole at $\lambda = -1/2$, but then we take the principal part (the lowest non-zero coefficient in the Laurent expansion).

2. One can multiply a distribution by a smooth function. Formally, the result is given by $f\xi(h) := \xi(fh)$. The corresponding operation on $\mathcal{D}_n$-modules is tensor product over $\mathcal{O}_n := \mathcal{O}(A^n) = k[x_1, \ldots, x_n]$. Note that a product of a smooth function and a generalized function (= functional on smooth measures) is a generalized function, a product of a function and a distribution is a distribution, and a product of a smooth measure and a distribution is not defined. Similarly, a product of two left $\mathcal{D}_n$-modules is a left $\mathcal{D}_n$-module, a product of a left $\mathcal{D}_n$-module by a right $\mathcal{D}_n$-module is a right $\mathcal{D}_n$-module, and a product of right $\mathcal{D}_n$-modules is not defined. The $\mathcal{D}_n$-module structure of a product of two (left) $\mathcal{D}_n$-modules is defined via Leibnitz rule: $\partial_i(m \otimes n) = \partial_i m \otimes n + m \otimes \partial_i n$. One can always turn a left $\mathcal{D}_n$-module to a right one using tensor product with the (right) $\mathcal{D}_n$-module of (algebraic) top differential forms.

3. For a polynomial map of affine spaces$\pi : X \to Y$, we can pullback smooth functions from Y to X. If the map is submersive then we can even pull generalized functions. Let us define pullback of $\mathcal{D}_n$-modules as well. Let M be an $\mathcal{O}_Y$-module. As an $\mathcal{O}_X$-module we define $\pi^0(M) := \mathcal{O}_X \otimes_{\mathcal{O}_Y} M$. The action of the vector fields $\mathcal{T}_Y$ is defined using the natural morphism $\mathcal{T}_X \to \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{T}_Y$, which on every fiber is defined using $d\pi$. In coordinates: $\xi(f \otimes m) = \xi(f) \otimes m + \sum_i f\xi(\pi^*(y_i)) \otimes \partial_i m$. By the well-known properties of pullback of $\mathcal{O}_X$-modules we get that $(\tau\pi)^0 = \pi^0\tau^0$, and that pullback is strongly right-exact, i.e. right-exact and commutes with arbitrary direct sums.

Exercise 4.1. Let A,B be rings. Let $F : \mathcal{M}(A) \to \mathcal{M}(B)$ be a strongly right-exact functor. Then $F(A)$ has a natural structure of an $A - B$-bimodule and F is isomorphic to the functor $M \mapsto F(A) \otimes_A M$.

Notation 4.2. $\mathcal{D}_{X\to Y} := \pi^0(\mathcal{D}_Y)$.

Remark 4.3. The intuition here is that $\mathcal{D}_{X\to Y}$ is the $(\mathcal{D}_X, \mathcal{D}_Y)$-bimodule of $\mathcal{O}_X$-valued differential operators on $\mathcal{O}_Y$. For a general commutative algebra A and an A-module M an $\leq n$-th degree differential operator on A with values in M is a $\mathbb{K}$-linear operator $D : A \to M$, such that $[a_1, [a_2, \ldots, [a_{n+1}, D]]] = 0$ for all $a_1, \ldots, a_{n+1} \in A$. Apparently, in nice cases $\mathcal{D}_{X\to Y}$, defined this way, coincides as an $(\mathcal{O}_X, \mathcal{D}_Y)$-bimodule with $\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{D}_Y$. From here it follows that for any $\mathcal{D}_Y$-module M we have $\mathcal{D}_{X\to Y} \otimes_{\mathcal{D}_Y} M \simeq \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \otimes_{\mathcal{D}_Y} M \simeq \mathcal{O}_X \otimes_{\mathcal{O}_Y} M$.

Remark 4.4. Functoriality amounts to the natural map $\mathcal{D}_{X\to Y} \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to Z} \to \mathcal{D}_{X\to Z}$ being an isomorphism. Since $\mathcal{D}_{X\to Y} \simeq \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{D}_Y$, this is automatic.

Theorem 4.5 (Bernstein)**.** *The pullback of a holonomic $\mathcal{D}_Y$-module is a holonomic $\mathcal{D}_X$-module.*

We divide the proof into several lemmas.

Lemma 4.6. *Any map $\pi : X \to Y$, where $Y \simeq \mathbb{A}^n$, can be decomposed into a standard embedding, an isomorphism, and a standard projection.*

Proof. Take the maps $X \to X \times Y \to X \times Y \to Y$, $x \mapsto (x, 0)$, $(x, y) \mapsto (x, y + \pi(x))$, $(x, y) \mapsto y$. □

Lemma 4.7. *Let X, Y be affine spaces. The pullback under the standard projection $p : T \times Y \to Y$ of a holonomic module is holonomic.*

Proof. In this case the pullback is the exterior product $\mathcal{O}_T \otimes_k M$. It is easy to see that exterior product of holonomic modules is holonomic. □

Lemma 4.8. *The pullback under an isomorphism $i : X \to Y$ of a holonomic $\mathcal{D}_Y$-module is a holonomic $\mathcal{D}_X$-module.*

Proof. In this case we can consider the pullback as the same space, just a different action. If F^iM is a good filtration for the original action and $r := \deg \pi$, then $\Phi^i M := F^{ri}M$is a filtration for the new action, and it satisfies $dim \Phi^i M \leq (cr^d)i^d$. □

Lemma 4.9. *The pullback under the standard embedding $i : X \to X \times \mathbb{A}^1$ of a holonomic $\mathcal{D}_{X\times\mathbb{A}^1}$-module is a holonomic $\mathcal{D}_X$-module.*

This lemma is the difficult one. Indeed, in this case the pullback of a finitely-generated module might be not finitely-generated. Indeed, for $X = pt$ we get $i^0(\mathcal{D}_X)$ To prove it, we will need another important lemma, that we in fact proved in the first lecture.

Lemma 4.10 (Kashiwara)**.** *Let N be a $\mathcal{D}_{X\times\mathbb{A}^1}$-module. Denote by t the coordinate of $\mathbb{A}^1$. Assume that t acts locally nilpotently on N and let $R := Kert$, $R_i := \partial_t^i R$. Then $N = \bigoplus_i R_i$ and $t\partial_t$ acts on $R_i$by the scalar $-(i + 1)$.*

Proof of Lemma 4.9. Let N be a holonomic $\mathcal{D}_{X\times\mathbb{A}^1}$-module. Denote by t the coordinate of $\mathbb{A}^1$. Then $M := i^0(N) = N/tN$. Denote by $N^0$the submodule consisting of elements annihilated by powers of t. Then, by Kashiwara's lemma, we have $tN_0 = N_0$. Thus $i^0(N) = i^0(N')$, where $N' = N/N_0$. Now, N' is also holonomic and t has no kernel on N'. Choose a good filtration F^iN' and define the corresponding good filtration F^iM by projection. Then $\dim F^iM \leq \dim F^iN' - \dim tF^{i-1}N' = \dim F^iN' - \dim F^{i-1}N' \leq ci^{\dim X}$. □

Theorem 4.5 follows now from Lemmas 4.6,4.7,4.8, and 4.9.

Corollary 4.11. *If $M, N \in \mathrm{Hol}\,(\mathcal{D}_X)$ then $M \otimes_{\mathcal{O}_X} N \in \mathrm{Hol}\,(\mathcal{D}_X)$.*

Proof. $M \otimes_X N = \Delta^0\,(M \otimes_{\mathbb{K}} N)$, $\Delta : X \to X \times X$ is the diagonal. □

4. For a polynomial map of affine spaces $\pi : X \to Y$, we can pushforward smooth compactly supported measures from Y to X, by integration by fibers.Note that we indeed push measures and not functions. This hints that the pushforward π_0 should be defined for right $\mathcal{D}_X$-modules.

Definition 4.12. For $M \in \mathcal{M}^r(\mathcal{D}_X)$ define $\pi_0(M) := M \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y} \in \mathcal{M}^r(\mathcal{D}_Y)$.

This operation also preserves holonomicity. This can again b shown by decomposing the map into three parts. The difficult case now will be the standard projection. However, we will prove this differently using a trick.

Exercise 4.13. Let $M \in \mathcal{M}^r(\mathcal{D}_V)$ and let $\mathcal{F}(M) \in \mathcal{M}^l(\mathcal{D}_{V^*})$ denote the module obtained from M by swapping the actions of $x_i$and ∂_i. Let $T : V \to W$ be a linear map, and let $T^* : W^* \to V^*$denote the dual map. Then $\mathcal{F}(T_0M) = (T^*)^0(\mathcal{F}(M))$.

Corollary 4.14. *Pushforward of a holonomic module is holonomic.*

Proof. For isomorphisms it is easy. The standard embeddings and projections are linear maps, and thus for them it follows from Exercise 4.13 and Theorem 4.5. □

Let us now examine how does pushforward look like. For $p : \mathbb{A}^1 \to pt$ we have $p_0(M) = M/M\partial_t$. For $i : pt \hookrightarrow \mathbb{A}^1$we have $i_0(k) := \bigoplus k\delta^i$, with $\delta^i\partial_t = \delta^{i+1}$and $\delta^i t = i\delta^{i-1}$.

Example 4.15. $p : \mathbb{A}^1 \to \{\mathrm{pt}\}$. $p_0(M) = M/M\partial_t$. For $i : \{\mathrm{pt}\} \to \mathbb{A}^1$: $i^0(M) := \bigoplus \mathbb{K}\delta^{(i)}$ is the $\mathcal{D}$-module of distributions supported at $\{\mathrm{pt}\}$. $\delta^{(i)}\partial_t := \partial^{(i+1)}$, $\delta^{(i)}t := i\delta^{(i-1)}$.

Lemma 4.16. *Let $\xi \in S^*(\mathbb{R}^n)$ be a tempered distribution, p be a positive polynomial, and $p \to \infty$ at ∞. Then $\lambda \mapsto \left\langle \xi, p^\lambda \right\rangle$ converges for $\Re\lambda < -r$ for some r.*

Lemma 4.17. *Let ξ be holonomic. Then there exist rational functions $q_1, \ldots, q_\ell \in \mathbb{C}(\lambda)$, such that*

$$\left\langle \xi, p^\lambda \right\rangle = \sum_i q_i \left\langle \xi, p^{\lambda-i} \right\rangle$$

Proof. Take the $\mathcal{D}$-module M generated by the distribution $p^{\lambda-k}\xi$ over the field$\mathbb{C}(\lambda)$. It is holonomic. Thus its pushfoward to the point is holonomic. On the other hand, the pushforward to the point is $M/\partial M$ ($\mathcal{D}_{X\to\mathrm{pt}} = \mathcal{O}(X)$). Being holonomic over a point means that it's a finite-dimensional vector space over $\mathbb{C}(\lambda)$. Thus $p^{\lambda-k}\xi$ are $\mathbb{C}(\lambda)$-linearly dependent modulo ∂M. The integral on ∂M vanishes, thus $\int \xi p^\lambda$ satisfies this linear dependence. □

5. Homological properties

Let $\mathcal{C}$ be an abelian category.

Definition 5.1. We say that $\mathcal{C}$ has homological dimension $\leq d$ if for any $M \in \mathcal{C}$ and any projective resolution $P_{d-1} \to \cdots \to P_0 \to M \to 0$, the kernel $\ker(P_{d-1} \to P_{d-2})$ is projective.

Theorem 5.2. *The following are equivalent for $\mathcal{C}$:*

1. *Any object has a projective resolution of length $\leq d$.*
2. Ext^{d+i} *vanishes for all $i \geq 1$.*
3. *The derived functor $L_{d+i}F$ vanishes for any right exact functor F and all $i \geq 1$..*
4. $\mathrm{hd}\,\mathcal{C} \leq d$.

Definition 5.3. Let V be a vector space, and let $a_1, \ldots, a_n : V \to V$ be commuting operators. The Koszul complex of $C(V, a_1, \ldots, a_n)$ is the complex (numbered by $n, \ldots, 0$)

$$0 \to \Lambda^n\mathbb{K}^n \otimes V \to \Lambda^{n-1}\mathbb{K}^n \otimes V \to \cdots \to \Lambda^0\mathbb{K}^n \otimes V \to 0$$

with differential $\sum_i \frac{\partial}{\partial\xi_i} \otimes a_i$, where $\frac{\partial}{\partial\xi_i}$ is the interior product with the basis vector ξ_i.

Definition 5.4. A sequence (a_i) is regular if a_i has no kernel on $V/(a_1V + \cdots + a_{i-1}V)$, for all i.

Theorem 5.5 (Proof-Exercise)**.** *If the sequence (a_i) is regular then the Koszul complex is acyclic outside* 0*, and $H^0(C) \simeq V/(a_1V + \cdots + a_nV)$.*

Let $A := \mathbb{K}[x_1, \ldots, x_n]$.

Theorem 5.6 (Hilbert's syzygy)**.** *The homological dimension of $A = \mathbb{K}[x_1, \ldots, x_n]$ is n.*

Proof. The Koszul complex of $x_1, \ldots, x_n$ acting on A is a free resolution of the module $A/(x_1, \ldots, x_n)A$. For an arbitrary module M let x_i act on $A \otimes_{\mathbb{K}} M$ by

$$x_i(a \otimes m) := x_i a \otimes m + a \otimes x_i m.$$

This defines an A-module structure on $A \otimes_{\mathbb{K}} M$. This module is free (exercise). Thus the complex

$$C(A, x_1, \ldots, x_n) \otimes_{\mathbb{K}} M$$

is a free resolution of M. □

Lemma 5.7 (Graded Nakayama's lemma)**.** *Let M be a finitely generated graded A-module with $M = (x_1, \ldots, x_n)M$. Then $M = 0$.*

Proof. Since M is f.g. and $M = (x_1, \ldots, x_n)M$, the Nakayama's lemma implies that $0 \in \operatorname{supp} M$. Since M is graded, $\operatorname{supp} M$ is conical and thus empty. □

Corollary 5.8. *Any graded projective finitely generated module P over $\mathbb{K}[x_1, \ldots, x_n]$ is free.*

Proof. Choose a basis $v_1, \ldots, v_m$ for $P/\mathfrak{m}P$, where $\mathfrak{m} := (x_1, \ldots, x_n)$. Lift v_i to homogeneous elements $p_i \in P$. By the graded Nakayama's lemma, p_i generate P. Thus we have a s.e.s. of graded modules $0 \to K \to A^m \to P \to 0$. Here A^m has its grading shifted according to the degrees of p_i. Since P is projective, this sequence splits. So $A^m \simeq K \oplus P$. Thus $K/\mathfrak{m}K = 0$, so $K = 0$. □

Corollary 5.9. *Any graded finitely generated A-module has a free graded resolution of length $\leq n$.*

Definition 5.10. For a Noetherian ring R we denote by $\mathcal{M}^f(R)$ the category of finitely-generated left R-modules, and by $\operatorname{hd}(R)$ the homological dimension of this category.

Exercise 5.11 (*)**.** $\operatorname{hd}(\mathcal{M}(R)) = \operatorname{hd}(R)$.

Exercise 5.12. Let R be a ring and $M \in \mathcal{M}^f(R)$ with a good filtration. Then

(i) for some l there exists a good filtration on R^l and a strict epimorphism $R^l \twoheadrightarrow M$.
(ii) If $\operatorname{Gr} M$ is free then M is free.

From Corollary 5.9 we obtain

Corollary 5.13. $\operatorname{hd} \mathcal{D}_n \leq 2n$.

Proof. Let $M \in \mathcal{M}^f(\mathcal{D}_n)$. Choose a good filtration on M. By Exercise 5.12(i) there exists a free $\mathcal{D}_n$-module F_1 with good filtration and a strict epimorphism $\varphi_1 : F_1 \twoheadrightarrow M$. Let L_1 be the kernel of φ_1 with induced filtration and choose a free F_2 again using Exercise 5.12(i). Continuing in this way we obtain an exact sequence of finitely-generated filtered modules with strict maps:

$$0 \to L_{2n-1} \to F_{2n-1} \to \cdots \to F_1 \to M \to 0,$$

with F_i free. By Exercise 1.20, the associated graded sequence

$$0 \to \operatorname{Gr} L_{2n-1} \to \operatorname{Gr} F_{2n-1} \to \cdots \to \operatorname{Gr} M \to 0$$

is also exact. By Hilbert's syzygy theorem, $\operatorname{Gr} L_{2n-1}$ is projective, and thus free. By Exercise 5.12(ii), L_{2n-1} is free and the above sequence is a free resolution of M of length $2n$. □

Now we want to show that $\operatorname{hd} \mathcal{D}_n = n$.

Corollary 5.14. *Let R be a Noetherian ring with* $\operatorname{hd} R < \infty$*, and M be a finitely generated R-module. Then* $\operatorname{hd} M \leq d$ *iff* $\operatorname{Ext}^i(M, R) = 0\,\forall i > d$.

Proof. Let $M \in \mathcal{M}^f(R)$ with $\operatorname{Ext}^i(M,R) = 0\,\forall i > d$. We have to show that $\operatorname{Ext}^i(M, \cdot) = 0\,\forall i > d$. Take any finitely generated X, and consider $0 \to L \to R^\ell \to X \to 0$. Thus:

$$\operatorname{Ext}^i\left(M, R^\ell\right) \to \operatorname{Ext}^i(M, X) \to \operatorname{Ext}^{i+1}(M, L) \to \operatorname{Ext}^{i+1}\left(M, R^\ell\right)$$

Thus for $i > d$ we have $\operatorname{Ext}^i(M,X) \simeq \operatorname{Ext}^{i+1}(M,L)$. By induction on i descending from $\operatorname{hd} M$, $\operatorname{Ext}^{>d}(M, \cdot) = 0$. □

Let $A := \mathbb{K}[x_1, \ldots, x_n]$, M be a f.g. A-module. Denote $E^i(M) := \operatorname{Ext}^i(M, A)$.

Theorem 5.15 (Serre ??)**.** *Let $d := d(M)$. Then $E^i(M) = 0, \forall i < n - d$.*

Proof. Induction on n. Take $B := \mathbb{K}[x_1, \ldots, x_{n-1}]$. If M is finitely generated over B, take

$$N := M[t] \simeq A \otimes_B M$$

Then $E^i(N) \simeq \operatorname{Ext}^i(M, B)[t]$. Thus $E^i(N) = 0$ for $i < n-1-d$, and $E^{n-1-d}(N)$ is free over $\mathbb{K}[t]$. Now we have a s.e.s.

$$0 \to N \overset{t-x_n}{\to} N \to M \to 0$$

Thus

$$E^{n-2-d}(N) \to E^{n-1-d}(M) \overset{0}{\to} E^{n-1-d}(N) \overset{t-x_n}{\to} E^{n-1-d}(N)$$

The rightmost map has trivial kernel, so the arrow in the middle is 0. Now $E^{n-2-d}(N) = 0$ implies $E^{n-1-d}(M) = 0$.

Now we treat the general case when M is not necessarily finitely generated over B. If $d(M) = n$ then there is nothing to prove, otherwise by Noether's normalization lemma there exists a linear coordinate change $y_i = Tx_i$ such that $A/\operatorname{Rad}(\operatorname{Ann}(M))$ is finite over $\mathbb{K}[y_1, \ldots, y_{n-1}]$. Then M is finitely generated over $\mathbb{K}[y_1, \ldots, y_{n-1}]$, and we reduce to the previous case. □

Lemma 5.16. *Assume that M is graded. Then*

$$d(M) \leq d(\operatorname{coker}(x_n|_M)) + 1.$$

Proof.

$$M^i \overset{x_n}{\to} M^{i+1} \to (\operatorname{coker}(x_n|_M))^{i+1} \to 0$$

Thus $\Delta d_M(i) \leq d_{\operatorname{coker} x_n}(i+1)$. □

Corollary 5.17. *Assume that* $\ker(x_n \restriction_M) = 0$*. Then*

$$d(M) \leq d(\operatorname{coker}(x_n|_M)) + 1.$$

Proof. $0 \to M \overset{x_n}{\to} M \to \operatorname{coker} x_n \to 0$. Introduce a filtration on M, pass to the associated graded module.

$$0 \to \operatorname{Gr} M \to \operatorname{Gr} M \to \operatorname{Gr} \operatorname{coker} x_n \to 0$$

Note that $\operatorname{Gr} \operatorname{coker} x_n = \operatorname{coker}(x_n \restriction_{\operatorname{Gr} M})$, and use the lemma on graded modules. □

Corollary 5.18. $d(M) \leq \max(d(\ker(x_n|_M)), d(\operatorname{coker}(x_n|_M)) + 1)$.

Proof. We can assume that $d(\ker(x_n \restriction_M)) < d(M)$. Then $d(\bigcup_i \ker(x_n^i|_M)) < d(M)$. Indeed, $\bigcup_i \ker(x_n^i \restriction_M)$ stabilizes at a finite union, and

$$\ker\left(x_n^i|_M\right)/\ker\left(x_n^{i-1}|_M\right) \simeq \ker\left(x_n|_{M/\ker\left(x_n^{i-1}\restriction_M\right)}\right),$$

whose d is $< d(M)$. Thus $d(M) = d(N)$, where

$$N := M/\bigcup_i \ker x_n^i$$

Now $\ker(x_n \restriction N) = 0$, so we reduce to the previous corollary. □

Lemma 5.19. $\operatorname{supp} E^i(M) \subset \operatorname{supp} M$

Proof. $\operatorname{Ann} M \subset \operatorname{Ann} E^i M$. □

Theorem 5.20 (Ross ??)**.** *For any $M \in \mathcal{M}^f(A)$, $d(E^i M) \leq n - i$.*

Proof. We prove by induction on n. Consider first the case when M is finitely generated over $B = \mathbb{K}[x_1, \ldots, x_{n-1}]$. $N := M[t] \simeq A \otimes_B M$, $E^i(N) = \operatorname{Ext}^i(M, B)[t]$ thus by the induction hypothesis

$$d\left(E^i(N)\right) \leq n - 1 - i + 1 = n - i$$

Now,

$$0 \to N \overset{t - x_n}{\to} N \to M \to 0$$

Thus

$$\cdots \to E^{i-1}(M) \overset{0}{\to} E^{i-1}(N) \to E^{i-1}(N) \to E^i(M) \to E^i(N) \to E^i(N) \to \ldots$$

The map $E^{i-1}(M) \to E^{i-1}(N)$ is 0 because $t - x_n$ has zero kernel. For any $v \in E^{i-1}(M)$, $(t - x_n)v = 0$, but $t - x_n$ has no kernel in $E^{i-1}(N)$ because $E^i(N) = \operatorname{Ext}^{i-1}(N, B)[t]$ and $t - x_n$ shifts the degree by 1.

Now introduce a filtration on $E^{i-1}(N)$ that is a grading in t. Then

$$0 \to F^j\left(E^{i-1}N\right) \to F^{j+1}\left(E^{i-1}N\right) \to F^{j+1}\left(E^i M\right) \to 0$$

Thus $d(E^i M) = d(E^{i-1}N) - 1 \leq n - (i-1) - 1 = n - i$.

The next case is that $x_n : M \to M$ is injective. Then

$$0 \to M \overset{x_n}{\to} M \to L \to 0$$

Thus

$$E^i L \to E^i M \overset{x_n}{\to} E^i M \to E^{i+1} L$$

By the last corollary, $d(E^i M) \leq \max(d(E^i L), d(E^{i+1} L) + 1) \leq n - i$.

Finally, in the general case

$$0 \to K \to M \to L \to 0$$

where x_n is nilpotent on K and $x_n \restriction_L$ is bijective Thus

$$\cdots \to E^i L \to E^i M \to E^i K \to \ldots$$

Thus $d(E^i M) \leq \max(d(E^i L), d(E^i K)) \leq n - i$ by the previous cases. □

Corollary 5.21 (Ross ??)**.** *Let $M \in \mathcal{M}^f(\mathcal{D}_n)$. Then*

(1) $\operatorname{Ext}^i(M, \mathcal{D}_n) = 0$, $\forall i < 2n - d(M)$

(2) $2n - d\left(\operatorname{Ext}^i(M, \mathcal{D}_n)\right) \geq i$

Proof. As we proved before, M has a resolution of length $2n$ consisting of free finitely generated filtered modules and strict maps:

$$0 \to F_{2n} \to \cdots \to F_0 \to 0$$

Taking Hom into $\mathcal{D}_n$ we get

$$0 \to F_0^* \to \cdots \to F_{2n}^* \to 0$$

Passing to associated graded we have

$$0 \to \mathrm{Gr}\, F_0^* \to \cdots \to \mathrm{Gr}\, F_{2n}^* \to 0$$

The cohomologies of the latter sequence are isomorphic both to $\mathrm{Ext}^i(\mathrm{Gr}\, M, A)$ and to $\mathrm{Gr}(\mathrm{Ext}^i(M, \mathcal{D}_n))$. The statements now follow from Theorems 5.15 and 5.20. □

Corollary 5.22. $\mathrm{hd}\, \mathcal{M}^f(\mathcal{D}_n) \le n$. *For any* $M \in \mathcal{M}^f(\mathcal{D}_n)$, $\mathrm{Ext}^n(M, \mathcal{D}_n)$ *is holonomic. For a holonomic module* $\mathrm{Ext}^{<n}(M, \mathcal{D}_n) = 0$.

Proof. $d(E^{n+i}M) \le n - i$, so by Bernstein's inequality, $E^{n+i}M = 0$ for $i > 0$. For $i = 0$ we get $d(E^n M) = n$. For holonomic M we have $n - i < 2n - d(M)$ and thus $\mathrm{Ext}^{n-i}(M, \mathcal{D}_n) = 0$ for any $i > 0$. □

Definition 5.23. Define $D : \mathrm{Hol}^\ell(\mathcal{D}_n) \to \mathrm{Hol}^r(\mathcal{D}_n)$ by $\mathrm{Ext}^n(\cdot, \mathcal{D}_n)$.

Theorem 5.24. *D is is an equivalence of categories, and* $D \circ D \simeq \mathrm{id}$.

Proof. To prove that $D \circ D \simeq \mathrm{id}$, take a free resolution of M:

$$0 \to F_n \to \cdots \to F_0 \to 0$$

and dualize it by $\mathrm{Hom}(\cdot, \mathcal{D}_n)$. Since M doesn't have smaller Ext's, this will be a free resolution of DM. Finally, $D \circ D \simeq \mathrm{id}$ implies that D is an equivalence of categories. □

Theorem 5.25. *For any* $M \in \mathcal{M}^f(\mathcal{D}_n)$ *there is a canonical embedding* $0 \to D(\mathrm{Ext}^n(M, \mathcal{D}_n)) \to M$*; moreover its image is the maximal holonomic submodule of* M.

Proof. $H = \mathrm{Ext}^n(M, \mathcal{D}_n)$. Let $0 \to P_n \to \cdots \to P_0 \to 0$ be a free resolution of M. Dualize it:

$$0 \to P_0^* \to \cdots \to P_n^* \to 0$$

Now consider a free resolution of H:

$$0 \to Q_n \to \cdots \to Q_0 \to 0$$

H is the last cohomology of P^*, so it is a factor of P_n^*. Now step by step we lift this to a map of complexes $P^* \to Q$. Dualizing, we get maps $Q^* \to P$. By Corollary 5.22, H is holonomic and thus $\mathrm{Ext}^{<n}(H, \mathcal{D}_n)$ vanish. Thus Q^* is a resolution of DH. Thus we get a map $DH \to M$ whose image is a holonomic submodule of M.

The map $DH \to M$ is injective because the right-exact functor $\mathrm{Ext}^n(\cdot, \mathcal{D}_n)$ maps it to the identity map. Finally, for any holonomic submodule $L \subset M$ we have an onto map $H \twoheadrightarrow DL$ and thus an embedding $L \subset DH$. □

Remark 5.26. Everywhere in this section we could have used the geometric filtration on $\mathcal{D}_n$ instead of the Bernstein filtration. Together with Theorem 5.21 this gives another proof of Theorem 3.13.

6. D-modules on smooth affine varieties

Let $X := \operatorname{Spec} A$ be an affine algebraic variety, and for $x \in X$ let $\mathbb{K}_x := A/\mathfrak{m}_x$. $T_x^* X := \mathfrak{m}_x/\mathfrak{m}_x^2$. We start with several well-known definitions and theorems from algebraic geometry.

Definition 6.1. Let $\operatorname{Der}\mathcal{O}(X)$ denote the algebra of derivations of $\mathcal{O}(X)$, i.e. linear endomorphisms of $\mathcal{O}(X)$ satisfying the Leibnitz rule. Elements of $\operatorname{Der}\mathcal{O}(X)$ are called (algebraic) vector fields on X.

Theorem 6.2. *The following are equivalent:*

(1) *$\mathbb{K}_x$ has finite homological dimension as an A-module.*
(2) *$\operatorname{Gr}_{\mathfrak{m}_x} A := \bigoplus (\mathfrak{m}_x^i/\mathfrak{m}_x^{i+1})$ is a polynomial algebra.*
(3) *$\dim T_x^* X = \dim_x X$.*
(4) *Locally around x there is a quasi-coordinate system.*

Definition 6.3. We say that X is smooth at x if these conditions hold.

Definition 6.4. A quasi-coordinate system of an affine variety U at $x \in U$ is:

(1) A collection of functions $x_1, \dots, x_n \in \mathcal{O}(U)$;
(2) a collection of vector fields $\partial_1, \dots, \partial_n \in \operatorname{Der}\mathcal{O}(U)$,

such that

(a) $\partial_i x_j = \delta_{ij}$;
(b) dx_i span $T_u^* X$ for all $u \in U$.

Theorem 6.5. *The set of smooth points is open and dense.*

Definition 6.6. $\mathcal{D}^{\leq -1}(X) := 0$,

$$\mathcal{D}^{\leq k}(X) := \left\{ d \in \operatorname{Hom}_{\mathbb{K}}(\mathcal{O}(X), \mathcal{O}(X)) \,\middle|\, \forall f \in \mathcal{O}(X) : [f, d] \in \mathcal{D}^{\leq k-1}(X) \right\}.$$

Similarly, for $\mathcal{O}(X)$-modules M, N define $\mathcal{D}^{\leq k}(M, N)$.

Example 6.7. $\mathcal{D}^{\leq 0}(X) = \mathcal{O}(X)$, $\mathcal{D}^{\leq 1}(X) = \mathcal{O}(X) \oplus \operatorname{Der}\mathcal{O}(X)$.

The algebra of algebraic differential operators is defined by $\mathcal{D}(X) := \bigcup_i \mathcal{D}^i(X)$. We will show that if X is smooth then $\mathcal{D}(X)$ is Noetherian and generated by $\mathcal{O}(X)$ and $\operatorname{Der}\mathcal{O}(X)$.

Exercise 6.8 (*)**.**

(1) If $X = \{\sum_i x_i^2 = 0\}$ then $\mathcal{D}(X)$ is Noetherian but not generated by $\mathcal{D}^{\leq 1}(X)$.
(2) If $X = \{\sum_i x_i^3 = 0\}$ then $\mathcal{D}(X)$ is not Noetherian.

From now on we assume that X is smooth.

Theorem 6.9. *Let $M, N \in \mathcal{M}(\mathcal{O}(X))$, let $d \in \mathcal{D}^{\leq k}(M, N)$ and $f \in \mathcal{O}(X)$. Then it uniquely defines $d' \in \mathcal{D}^{\leq k}(M_f, N_f)$.*

Proof. Define $d(f^{-i}m)$ by induction on i and k:

$$d'\left(f^{-i}m\right) := f^{-1}d'\left(f^{-i+1}m\right) - f^{-1}[d', f]\left(f^{-i}m\right)$$

□

Corollary 6.10. *If M is finitely generated then $(\mathcal{D}(M, N))_f \simeq \mathcal{D}_{\mathcal{O}_f}(M_f, N_f)$.*

Proof. To construct the map in the only nontrivial direction $\mathcal{D}_{\mathcal{O}_f}(M_f, N_f) \to (\mathcal{D}(M,N))_f$, take the common denominator of the generators of M. □

Definition 6.11. Define the sheaf $\mathcal{D}_X$ of differential operators on X by $\mathcal{D}_X(X_f) := \mathcal{D}(X_f)$.

By Corollary 6.10, $\mathcal{D}_X$ is a quasi-coherent sheaf.

Remark 6.12. In general, a good calculus of fractions is guaranteed by the Ore condition. For a ring A and a multiplicative set S, the Ore condition is that for any $a \in A, s \in S$ there are $a' \in A, s' \in S$, such that $as' = sa'$ (i.e. $s^{-1}a = a's'^{-1}$). For $S = \{f^n\}, f \in \mathcal{O}_X$, $A = \mathcal{D}_X$, it is satisfied.

Recall that τ_X denotes the tangent sheaf of X. Note that the existence of a quasi-coordinate system implies that τ_X is coherent and locally free.

Theorem 6.13. $\Sigma := \operatorname{Gr}\mathcal{D}(X) \simeq \mathcal{O}(T^*X) := \operatorname{Sym}\tau_X(X) \simeq \operatorname{Sym}\left(\mathcal{D}^{\leq 1}(X)/\mathcal{O}(X)\right)$.

Proof. Define $\Sigma^\ell := \operatorname{Sym}^\ell \tau_X(X)$. For $d \in \mathcal{D}^{\leq \ell}$ consider its symbol

$$(\sigma d)(f_1, \dots, f_\ell) := [[d, f_1], \dots, f_\ell].$$

The form σd is symmetric in f_i, and is a derivation in each f_i. Thus σd is an element of $\operatorname{Sym}^\ell(\tau_X(X))$.

Clearly, $d \mapsto \sigma d$ is an embedding. To show that it is onto, just take the product of vector fields to produce a given symbol. □

Remark 6.14. Incidentally, this also proves that vector fields generate $\operatorname{Der}\mathcal{O}(X)$ locally, i.e. as a sheaf. Since on affine varieties sheaf cohomology vanishes, this is also true globally.

Corollary 6.15. *$\mathcal{D}(X)$ is Noetherian and $hd(\mathcal{D}(X)) \leq 2\dim X$.*

Exercise 6.16. The structure of a left $\mathcal{D}(X)$-module on an $\mathcal{O}(X)$-module M is the same as the structure of a τ_X-module on M satisfying

$$(f\xi)m = f(\xi m) \text{ and } \xi(fm) - f(\xi m) = \xi(f)m.$$

The structure of a right $\mathcal{D}(X)$-module on an $\mathcal{O}(X)$-module M is the same as the structure of a module over the opposite of the Lie algebra τ_X satisfying

$$(f\xi)m = f(\xi m) \text{ and } \xi(fm) - f(\xi m) = -\xi(f)m.$$

By module over the opposite Lie algebra we mean the identity $\xi_1(\xi_2 m) - \xi_2(\xi_1 m) = -[\xi_1, \xi_2]m$.

Exercise 6.17. The module of top differential forms Ω_X^{top} with the action $\xi\alpha := -Lie_\xi\alpha$ (Lie derivative) is a right $\mathcal{D}(X)$-module. Moreover, $M \mapsto M \otimes_{\mathcal{O}_X} \Omega_X^{top}$ defines an equivalence of categories $\mathcal{M}(\mathcal{D}(X)) \simeq \mathcal{M}^r(\mathcal{D}(X))$.

The push and pull functors are defined for affine varieties in the same way as for affine spaces. Namely, for $\pi : X \to Y$ and $N \in \mathcal{M}(\mathcal{D}(Y))$ define $\pi^0(N) := \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} N$, with the action of $\tau(X)$ given by the morphism $\tau(X) \to \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \tau(Y)$. As before, π^0 is strongly right-exact and thus $\pi^0(N) = \mathcal{D}_{X\to Y} \otimes_{\mathcal{D}(Y)} N$, where $\mathcal{D}_{X\to Y} = \pi^0(\mathcal{D}(Y)) = \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathcal{D}(Y)$. For $M \in \mathcal{M}(\mathcal{D}(X))$ we define $\pi_0(M) := M \otimes_{\mathcal{D}(X)} \mathcal{D}_{X\to Y}$.

7. $\mathcal{D}$-modules on general separated smooth varieties

Fact 7.1. *For a variety TFAE:*

(i) For any affine U, V the intersection $U \cap V$ is affine, and $\mathcal{O}(U) \otimes_{\mathbb{K}} \mathcal{O}(V) \to \mathcal{O}(U \cap V)$ is onto.
(ii) There is an open affine covering (U_i), s.t. the previous property holds for each U_i, U_j
(iii) $\Delta X \subset X \times X$ is closed.

Varieties that satisfy these properties are called separated.

Definition 7.2. Let X be a smooth separated variety. Define the quasi-coherent sheaf of $\mathcal{O}_X$-algebras $\mathcal{D}_X$ by the property $\mathcal{D}_X(U) = \mathcal{D}(U)$ for every open affine $U \subset X$.

A $\mathcal{D}_X$-module is a sheaf of modules over the sheaf of algebras $\mathcal{D}_X$ that is quasi-coherent as a sheaf of $\mathcal{O}_X$-modules. That is, it's a quasi-coherent sheaf $\mathcal{F}$, such that $\mathcal{F}(U)$ have compatible structures of $\mathcal{D}_X(U)$-modules. We will denote the category of $\mathcal{D}_X$-modules by $\mathcal{M}(\mathcal{D}_X)$ and the category of quasi-coherent sheaves by $\mathcal{M}(\mathcal{O}_X)$.

Serre's theorem implies that for an affine X, $\mathcal{M}(\mathcal{D}_X) \simeq \mathcal{M}(\mathcal{D}(X))$.

Definition 7.3. A morphism of algebraic varieties $\pi : X \to Y$ is called affine if $\pi^{-1}(U)$ is affine for any open affine $U \subset Y$.

Example 7.4. Open and closed embeddings are affine.

Definition 7.5. For an affine morphism $\pi : X \to Y$, define the functors π^0 and π_0 gluing from affine pieces. In other words, $\pi^0(\mathcal{G})(\pi^{-1}(U)) := (\pi|_{\pi^{-1}(U)})^0(\mathcal{G}(U)$ and $\pi_0(\mathcal{F})(U) := (\pi|_{\pi^{-1}(U)})_0(\mathcal{F}(\pi^{-1}(U))$ for any open affine $U \subset Y$.

Example 7.6. Let $i_0 : Z \to X$ be a closed embedding of a smooth subvariety. One can choose local coordinates x_i, such that Z is given by $x_{m+1} = \cdots = x_n = 0$, $\mathcal{D}(X) \simeq \mathcal{O}(X) \otimes_{\mathbb{K}} \mathbb{K}[\partial_1, \ldots, \partial_n]$, $\mathcal{O}(Z) \simeq \mathcal{O}(X)/J$, $J := \langle x_{m+1}, \ldots, x_n \rangle$. Then $i^0\mathcal{F} = \mathcal{F}/J$.

Exercise 7.7. Let $V \subset X$ be an open subset and let $i : V \hookrightarrow X$ denote the embedding. Then

(i) $i_0(\mathcal{F})(U) = \mathcal{F}(V \cap U)$ for any $\mathcal{F} \in \mathcal{M}(\mathcal{D}_V)$ and any open $U \subset X$.
(ii) The functor $i_0 : \mathcal{M}(\mathcal{D}_U) \to \mathcal{M}(\mathcal{D}_X)$ is right-adjoint to the restriction functor $Res_U : \mathcal{M}(\mathcal{D}_X) \to \mathcal{M}(\mathcal{D}_U)$.
(iii) The functors i_0 and Res_U are exact.

Fact 7.8. *For a coherent sheaf TFAE:*

(i) It is locally free
(ii) It is projective
(iii) The dimension of the fiber is locally constant.

For non-affine X, the categories of $\mathcal{O}_X$-modules and $\mathcal{D}_X$-modules do not have enough projectives, but:

Fact 7.9. *$\mathcal{M}(\mathcal{O}_X)$ and $\mathcal{M}(\mathcal{D}_X)$ have enough injectives.*

Proof. Let us show this for $\mathcal{D}_X$-modules, since the proof for $\mathcal{O}_X$-modules is identical.

First we prove for affine X. For a projective right $\mathcal{D}_X$-module P the module $\operatorname{Hom}_{\mathbb{K}}(P, \mathbb{K})$ is an injective left $\mathcal{D}_X$-module. For any projective P and $P \to \operatorname{Hom}_{\mathbb{K}}(M, \mathbb{K}) \to 0$ we have embeddings

$$M \hookrightarrow \operatorname{Hom}_{\mathbb{K}}(\operatorname{Hom}_{\mathbb{K}}(M, \mathbb{K}), \mathbb{K}) \hookrightarrow \operatorname{Hom}_{\mathbb{K}}(P, \mathbb{K}).$$

For non-affine varieties, choose a finite open affine cover $X = \bigcup_j U_j$, and consider $i_0 : \mathcal{M}\left(\mathcal{D}_{U_j}\right) \to \mathcal{M}(\mathcal{D}_X)$. The functor i_0 is exact and maps injective sheaves to injective ones. Since $\mathcal{F} \restriction_{U_j}$ embeds into injective Q_j, $\mathcal{F}$ embeds into $\bigoplus_j i_{j*}Q_j$. □

Definition 7.10. A $\mathcal{D}_X$-module is called coherent if it is locally finitely generated.

Recall that for an affine variety X, $\operatorname{Gr}\mathcal{D}(X) = \mathcal{O}(T^*X)$.

Definition 7.11. For $\mathcal{F} \in \mathcal{M}_{\mathrm{coh}}(\mathcal{D}_X)$ choose a good filtration on $\mathcal{F}$, and define

$$\operatorname{Sing\,supp}_U F := \operatorname{supp}\operatorname{Gr}\mathcal{F}(U) \subset T^*X, \text{ and } \operatorname{Sing\,supp}\mathcal{F} := \bigcup_U \operatorname{Sing\,supp}_U \mathcal{F}.$$

By Theorems 3.3 and 3.5 this does not depend on the choice of a good filtration on $\mathcal{F}$.

Theorem 3.16 holds for singular support as well, though we won't prove it.

Theorem 7.12 (Kashiwara-Kawai-Sato, Gabber)**.** *For any $\mathcal{F} \in \mathcal{M}_{\mathrm{coh}}(\mathcal{D}_X)$,* Sing supp $\mathcal{F}$ *is a coisotropic subvariety of T^*X.*

This implies the Bernstein inequality, namely $\dim \operatorname{Sing\,supp}\mathcal{F} \geq \dim X$ if $\mathcal{F} \neq 0$. Another way of proving the Bernstein inequality is to reduce it to affine varieties, then to affine spaces, then use Proposition 3.13 to reduce to the classical Bernstein inequality for the arithmetic filtration (Theorem 1.31 above).

However, we are going to give a direct proof of the Bernstein inequality in the next section.

8. Kashiwara's lemma and its corollaries

Let $Z \subset X$ be a closed smooth subvariety and let $i : Z \hookrightarrow X$ denote the embedding. Let $\mathcal{M}^{\mathrm{r}}_Z(\mathcal{D}_X)$ denote the category of right $\mathcal{D}_X$-modules supported at Z. Our goal in this section is to prove and use the following theorem.

Theorem 8.1 (Kashiwara)**.** *The functor i_0 is an equivalence $\mathcal{M}^{\mathrm{r}}(\mathcal{D}_Z) \simeq \mathcal{M}^{\mathrm{r}}_Z(\mathcal{D}_X)$.*

For the proof we will need some constructions and lemmas.

Definition 8.2. Define $i' : \mathcal{M}^{\mathrm{r}}(\mathcal{D}_X) \to \mathcal{M}^{\mathrm{r}}(\mathcal{D}_Z)$ by

$$i'(\mathcal{F}) := \operatorname{Hom}_{\mathcal{D}_X}(\mathcal{D}_{Z\to X}, \mathcal{F}).$$

For $\mathcal{F} \in \mathcal{M}^r(\mathcal{D}_X)$ define $\Gamma_Z(\mathcal{F})(U) := \{\xi \in \mathcal{F}(U) \mid \operatorname{supp}\xi \subset Z\}$.

Exercise 8.3. (i) For affine X, $i'(M) \simeq \operatorname{Ann}_M I(Z)$.
(ii) $i' i_0 \mathcal{H} \simeq \mathcal{H}$ for any $\mathcal{H} \in \mathcal{M}(\mathcal{D}_Z)$.
(iii) i_0 is left adjoint to i'.

From the adjunction, we have a counit map $i_0 i' \mathcal{F} \to \mathcal{F}$.

Lemma 8.4. *Let $M \in \mathcal{M}(\mathcal{D}_1)$. Assume $M = \bigcup_i \ker x^i$. Then $M = \bigoplus_i \partial^i \ker x$.*

Proof. 1. We note that $(\partial x + i)\,\partial^i(\ker x) = 0$. Indeed, $[x, \partial^i] = -i\partial^{i-1}$, so $(\partial x + i)\,\partial^i = \partial^{i+1}x$.

2. $x\partial^i(\ker x) = -i\partial^{i-1}(\ker x)$. Thus $\partial^i \ker x \subset \ker x^{i+1}$.

3. $(\partial x + i)\ker x^{i+1} \subset \ker x^i$. Indeed, for $m \in \ker x^{i+1}$, $x(\partial x + i)\,m = (i-1)\,xm + \partial x^2 m = (\partial x + i - 1)\,xm \subset \ker x^{i-1}$ by induction.

4. $\partial^i \ker x$ are the different eigenspaces of ∂x, so the sum is direct. Now we show that

$$\ker x^i = \bigoplus_{j=0}^{i-1} \partial^j \ker x$$

Take $m \in \ker x^{i+1}$. Them $(\partial x + i)\, m \in \ker x^i$, so by induction $(\partial x + i)\, m \in \bigoplus_{j=0}^{i-1} \partial^j \ker x$. Again, by induction, $xm \in \bigoplus_{j=0}^{i-1} \partial^j \ker x$, so $\partial xm \in \bigoplus_{j=1}^{i} \partial^j \ker x$. Thus $m \in \bigoplus_{j=0}^{i} \partial^j \ker x$. □

Example 8.5. Distributions on $\mathbb{R}$ supported at 0 are sums of derivatives of the δ-function.

Lemma 8.6 (Standard in algebraic geometry)**.** *For any $x \in Z$, there is an open neighborhood $U \subset X$ and a quasi-coordinate system x_i on U, such that $Z \cap U$ is given by $x_{m+1} = \cdots = x_n = 0$, and* $\det(\partial x_i) \neq 0$.

Theorem 8.7. *$\varphi : i_0 i' \mathcal{F} \to \Gamma_Z(\mathcal{F})$ is an isomorphism.*

Proof. It's enough to show this locally. Choose a quasi-coordinate system on X (or an open subset of it), such that $Z = \{x_{m+1} = \cdots = x_n = 0\}$. We can assume $i = m+1$ by induction (locally there is a flag of smooth subvarieties, constructed using x_i). The induction step will be for $Z \subset Y \subset X$:

$$(Z \to Y)_0 (Y \to X)_0 (Y \to X)' (Z \to Y)' \mathcal{F} \simeq (Z \to Y)_0 \Gamma_Y (Z \to Y)' \mathcal{F} \simeq$$

$$\simeq (Z \to Y)_0 (Z \to Y)' \Gamma_Y \mathcal{F} \simeq \Gamma_Z \Gamma_Y \mathcal{F} \simeq \Gamma_Z \mathcal{F}$$

The nontrivial equality here follows by noting that Γ_Y consists of sections supported at Y, and $(Z \to Y)'$ consists of the sections killed by $I(Z)$.

Finally, define $Z := \{x_n = 0\}$. Let M be a right $\mathcal{D}(X)$-module, and $N := \Gamma_Z(M)$.

$$i' M = \ker(x_n \restriction_M)$$

and $i_0 i' M = \bigoplus_j i' M\, \partial^j_{x_n}$ because $\mathcal{D}_{Z \to X} \simeq \mathcal{O}_Z \otimes_{\mathcal{O}_X} \mathcal{D}_X \simeq \mathcal{D}_X / x_n \mathcal{D}_X \simeq \mathcal{D}_Z \otimes_{\mathbb{K}} \mathbb{K}[\partial_{x_n}]$. By Lemma 8.4,

$$\bigoplus_j \ker(x_n \restriction_N)\, \partial^j_{x_n} \simeq N$$

□

Corollary 8.8 (Kashiwara)**.** *The functors i_0 and i' define an equivalence of categories*

$$\mathcal{M}^{\mathrm{r}}(\mathcal{D}_Z) \simeq \mathcal{M}^{\mathrm{r}}_Z(\mathcal{D}_X).$$

8.1. Corollaries.

Lemma 8.9 (Exercise)**.** *For $\mathcal{H} \in \mathcal{M}_{\mathrm{coh}}(\mathcal{D}_Z)$, $i_0 \mathcal{H} \in \mathcal{M}_{\mathrm{coh}}(\mathcal{D}_X)$ and*

$$\operatorname{Sing\,supp}(i_0 \mathcal{H}) = \{(x, \xi) \in T^* X \mid (x, p_x \xi) \in \operatorname{Sing\,supp} \mathcal{H}\},$$

where $p_x : (T_x^X)^* \twoheadrightarrow (T_x^Z)^*$ is the dual map to the embedding $T_x Z \hookrightarrow T_x X$.

Corollary 8.10 (Bernstein's inequality)**.** *For any $\mathcal{F} \in \mathcal{M}_{\mathrm{coh}}(\mathcal{D}_X)$,* $\dim \operatorname{Sing\,supp} \mathcal{F} \geq \dim X$.

Proof. Let $\mathcal{F} \in \mathcal{M}^{\mathrm{r}}_{\mathrm{coh}}(\mathcal{D}_X)$. Suppose that $\dim \operatorname{Sing\,supp} \mathcal{F} < \dim X$. Let $p_X : T^*X \to X$ be the canonical projection. Let $Z := \overline{(p_X(\operatorname{Sing\,supp} \mathcal{F}))} \subsetneq X$. Then $\dim Z < \dim X$. There is $U \subset X$, such that $Z' := U \cap Z$ is nonsingular (and nonempty). $\mathcal{F}' := \mathcal{F}\restriction_U$. Then $\operatorname{supp} \mathcal{F}' \subset Z'$. By Kashiwara's lemma, $\mathcal{F}' \simeq i_0 i' \mathcal{F}'$, where $i : Z' \to U$. By induction hypothesis, $\dim \operatorname{Sing\,supp} i'\mathcal{F}' \geq \dim Z'$. Thus

$$\dim \operatorname{Sing\,supp} i_0 i' \mathcal{F}' \geq \dim \operatorname{Sing\,supp} i'\mathcal{F}' + \dim U - \dim Z' \geq \dim X.$$

But $\dim \operatorname{Sing\,supp} \mathcal{F}' < \dim X$ by assumption. This leads to a contradiction. □

Lemma 8.11. *Let $\mathcal{F} \in \mathcal{M}_{\mathrm{coh}}(\mathcal{D}_X)$. TFAE:*

(1) $\operatorname{Sing\,supp} \mathcal{F} \subset X \times \{0\} \subset T^*X$
(2) $\mathcal{F} \in \mathcal{M}_{\mathrm{coh}}(\mathcal{O}_X)$
(3) $\mathcal{F}$ is locally free of finite rank over $\mathcal{O}_X$

Proof. $3 \Rightarrow 2$ is obvious, $2 \Rightarrow 1$ is obvious (just take the generators over $\mathcal{O}_X$ and use them to construct a good filtration).

$1 \Rightarrow 2$: Choose local coordinates in an open affine $U \subset X$. Let $M = \mathcal{F}(U)$. Choose generators $v_1, \ldots, v_n$ of M over $\mathcal{D}_U$. Then we assume that $Z(\sigma \operatorname{ann}\{v_i\}) = U \times \{0\}$ (where σ is the symbols, and Z is the variety of zeros). Then for any i, j there is ℓ_{ij}, such that

$$\partial_j^{\ell_{ij}} v_i \in \mathcal{D}_U^{<\ell_{ij}} \{v_1, \ldots, v_n\}$$

Let $S := \left\{ \partial_1^{\ell_1} \ldots \partial_m^{\ell_m} v_i \,\middle|\, \ell_j < \ell_{ij} \right\}$. Then this set generates $F(U)$ over $\mathcal{O}_U$, so F is coherent over $\mathcal{O}_X$.

$2 \Rightarrow 3$: We can assume that X is affine. Let $\ell := \min_x \dim \mathcal{F}_x$. Then there is some $U \subset X$, such that $\dim \mathcal{F}_x = \ell$ for all $x \in U$ (we assume that X is connected and irreducible). Suppose $U \neq X$, i.e. there is $x \in X$, such that $\dim F_x > \ell$. Then there is a smooth affine curve $\nu : C \to X$ passing through x (cf. curve selection lemma), such that all the other points of this curve are in U. Take the $\mathcal{D}_C$-module $\nu^0 F$. It's $\mathcal{O}_C$-coherent because it's the pullback of $\mathcal{O}$-modules, and this operation preserves coherence. On the other hand, $\mathcal{O}_C$ is a Dedekind domain, so since $M := \nu^0 F$ is not locally free, it must have torsion. The torsion part M^{tor} is also a $\mathcal{D}_C$-module, and it has finite support $i : Z \subset C$. Thus $M^{\mathrm{tor}} = i_0 V$ for some $\mathcal{D}_Z$-module V. But $i_0 V$ is not finitely generated over $\mathcal{O}_C$ unless $V = 0$. □

Definition 8.12. $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules are called smooth.

Corollary 8.13. *Let $\mathcal{F}$ be a holonomic $\mathcal{D}_X$-module. Then there exists an open dense $U \subset X$, such that $\mathcal{F}\restriction_U$ is smooth (possibly trivial).*

Proof. $\operatorname{Sing\,supp} \mathcal{F}$ is n-dimensional, so it consists of a part of the form $U \times \{0\}$ and something else that projects to a lower-dimensional subvariety of X. □

Definition 8.14. For a closed subvariety $X \subset \mathbb{A}^n$ define the category of $\mathcal{D}_X$-modules as the category of $\mathcal{D}_{\mathbb{A}^n}$-modules supported at X.

Theorem 8.15. *This definition doesn't depend on the embedding.*

Proof. Let $\nu : X \to \mathbb{A}^n$, $\mu : X \to \mathbb{A}^m$. Take the embedding $\nu \times \mu : X \to \mathbb{A}^{n+m}$. Then there is $\rho : \mathbb{A}^n \to \mathbb{A}^m$, such that $\mu = \rho\nu$. Thus we have a closed embedding $i := \mathrm{id} \times \rho : \mathbb{A}^n \to \mathbb{A}^{n+m}$. Then $\mathcal{M}_X(\mathcal{D}_{\mathbb{A}^n}) \overset{i_0}{\simeq} \mathcal{M}_X(\mathcal{D}_{\mathbb{A}^{n+m}})$. □

Definition 8.16. Define $\mathcal{D}$-modules on general varieties by gluing affine ones. Note that for affine ones the notion is local.

For $F \in \mathcal{M}_X(\mathcal{D}_{\mathbb{A}^n})$ define $\operatorname{Sing\,supp}_X(F) := p_X(\operatorname{Sing\,supp} F)$, where $p_X : T^*\mathbb{A}^n \restriction_X \to T^*X$.

9. $\mathcal{D}$-modules on the projective space

$$V \overset{j}{\supset} V^\times \overset{p}{\to} \mathbb{P}(V)$$

For an $\mathcal{O}_{\mathbb{P}(V)}$-module $\mathcal{F}$ define

$$p^*\mathcal{F} := \mathcal{O}_{V^\times} \otimes_{\mathcal{O}_{\mathbb{P}(V)}} \mathcal{F}, \quad \tilde{\mathcal{F}} := j_* p^* \mathcal{F}.$$

There is an action of $\mathbb{G}_m$ on $\tilde{\mathcal{F}}$ by dilation, so it defines a grading on the global sections $\Gamma\left(\tilde{\mathcal{F}}\right)$, and $\Gamma(\mathcal{F}) = \left(\Gamma\left(\tilde{\mathcal{F}}\right)\right)^0$.

Let $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ be an exact sequence of $\mathcal{D}_{\mathbb{P}(V)}$-modules. Then $p^0\mathcal{F}_1 \to p^0\mathcal{F}_2 \to p^0\mathcal{F}_3$ is exact. While $\tilde{\mathcal{F}}_1 \to \tilde{\mathcal{F}}_2 \to \tilde{\mathcal{F}}_3$ may not be exact, the homology $\mathcal{H}$ is supported at 0. Thus $\mathcal{H} \simeq i_0 L$, where L is a vector space, and $i : \{0\} \hookrightarrow V$.

Let $E := \sum x_i \partial_i \in \mathcal{D}(V)$ be the Euler operator.

Exercise 9.1. On $\Gamma(i_0 L)$, E has negative eigenvalues.

$(\Gamma(i_0L))^0 = 0$, thus $(\Gamma(\mathcal{H}))^0 = 0$, so

$$\left(\Gamma\left(\tilde{\mathcal{F}}_1\right)\right)^0 \to \left(\Gamma\left(\tilde{\mathcal{F}}_2\right)\right)^0 \to \left(\Gamma\left(\tilde{\mathcal{F}}_3\right)\right)^0$$

is exact. On the other hand, $\left(\Gamma\tilde{\mathcal{F}}_i\right)^0 \simeq \Gamma\mathcal{F}_i$. Thus:

Lemma 9.2. *The functor of global sections*

$$\Gamma_{\mathbb{P}(V)} : \mathcal{M}(\mathcal{D}_{\mathbb{P}^n}) \to \mathcal{M}(\Gamma(\mathcal{D}_{\mathbb{P}^n}))$$

is exact.

Exercise 9.3. $\mathcal{D}_{\mathbb{P}^n} \simeq \mathcal{D}_n^0/\mathcal{D}_n^0 E$, where $\mathcal{D}_n^0$ is the zero-part of the grading on $\mathcal{D}_n$ given by the commutator with the Euler vector field. In other words, $\deg x_i = 1$, $\deg \partial_i = -1$.

Exercise 9.4. For any graded $\mathbb{K}[x_0, \dots, x_n]$-module we define a quasicoherent sheaf on the projective space M' by $M'(U) := \left(M\left(P^{-1}(U)\right)\right)^0$, where $P : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is the canonical projection. Any quasicoherent sheaf on $\mathbb{P}^n$ is obtained this way. More precisely,

$$\mathcal{M}^{\mathrm{qc}}(\mathcal{O}_{\mathbb{P}^n}) \simeq \mathcal{M}^{\mathrm{qc}}(\mathcal{O}_{\mathbb{A}^n}) / \mathcal{M}^{\mathrm{qc}}_{\{0\}}(\mathcal{O}_{\mathbb{A}^n})$$

(quotient w.r.t. a Serre subcategory).

Hint. $\mathbb{K}[x_0, \dots, x_n] \simeq \bigoplus_{d \geq 0} \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$, where $\mathcal{O}_{\mathbb{P}^n}(d)$ is the sheaf on $\mathbb{P}^n$ obtained by shifting by d the grading in the graded module $\mathbb{K}[x_0, \dots, x_n]$ (alternative description: $\mathcal{O}_{\mathbb{P}^n}(-1)$ is the canonical line bundle, $\mathcal{O}_{\mathbb{P}^n}(1)$ is its dual, and $\mathcal{O}_{\mathbb{P}^n}(d_1 + d_2) \simeq \mathcal{O}_{\mathbb{P}^n}(d_1) \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(d_2)$).

Take a sheaf $\mathcal{F}$, and the module $M_{\mathcal{F}} := \bigoplus_{d \geq 0} \Gamma(\mathbb{P}^n, \mathcal{F}(d))$, where $\mathcal{F}(d) := \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(d)$. Now take the sheaf $M'_{\mathcal{F}}$ corresponding to M_F. We claim that $M'_{\mathcal{F}} \simeq \mathcal{F}$. After that we prove that the kernel of the functor $M \mapsto \mathcal{F}$ consists of the sheaves supported at 0. $\square$

Lemma 9.5. $\Gamma_{\mathbb{P}(V)} : \mathcal{M}\left(\mathcal{D}_{\mathbb{P}^n}\right) \to \mathcal{M}\left(\Gamma\mathcal{D}_{\mathbb{P}^n}\right)$ *is faithful.*

Proof. Since $\Gamma_{\mathbb{P}(V)}$ is exact, it is enough to show that $\Gamma\left(\mathcal{F}\right) \neq 0$ for $\mathcal{F} \neq 0$.

Let j be such that $\operatorname{supp} M^j \not\subset \{0\}$ and $\operatorname{supp} M^\ell \subset \{0\}$ $\forall l$ with $|\ell| < j$. We want to show that $j = 0$. Suppose first that $j < 0$ and let $\xi \in M^j$, such that $\operatorname{supp}\xi \not\subset \{0\}$. Then there is $0 \leq i \leq n$, such that $\operatorname{supp} x_i\xi \not\subset \{0\}$. But $x_i\xi \in M^{j+1}$, so this contradicts our assumption. Similarly, for $j > 0$, take $\xi \in M^j$. $j\xi = E\xi = \sum_i x_i\partial_i\xi$, so there is i, such that $\operatorname{supp}\partial_i\xi \not\subset \{0\}$. But $\partial_i\xi \in M^{j-1}$, so again we get a contradiction. ☐

Lemma 9.6. $\operatorname{Hom}\left(\mathcal{D}_{\mathbb{P}^n}, \mathcal{F}\right) \simeq \Gamma\left(\mathcal{F}\right)$.

Proof. The internal Hom is $\mathcal{F}$, so the categorical Hom consists of its global sections. ☐

Corollary 9.7 (Bernstein-Beilinson, ??)**.** $\mathcal{D}_{\mathbb{P}^n}$ *is a projective generator of* $\mathcal{M}\left(\mathcal{D}_{\mathbb{P}^n}\right)$, *and thus* $\Gamma : \mathcal{M}\left(\mathcal{D}_{\mathbb{P}^n}\right) \to \mathcal{M}\left(\Gamma(\mathcal{D}_{\mathbb{P}^n})\right)$ *is an equivalence.*

Theorem 9.8 (Bernstein-Beilinson, ??)**.** $\Gamma(\mathcal{D}_{\mathbb{P}^n}) = \mathcal{D}_{\mathbb{P}^n}\left(\mathbb{P}^n\right) \simeq \mathcal{D}^0_{n+1}/E\mathcal{D}^0_{n+1}$, *where* $\mathcal{D}^0_{n+1}$ *is according to the grading* $\deg x_i = 1, \deg\partial_i = -1$, *and E is the Euler field.*

For the proof we will need some lemmas.

Exercise 9.9. There is a natural map $\mathcal{D}^0_{n+1}/E\mathcal{D}^0_{n+1} \to \mathcal{D}_{\mathbb{P}^n}\left(\mathbb{P}^n\right)$.

Exercise 9.10. $\operatorname{Gr}\left(\mathcal{D}^0_{n+1}/E\mathcal{D}^0_{n+1}\right) \simeq \mathcal{O}_{(T^*\mathbb{P}^n}\left(T^*\mathbb{P}^n\right)$.

Lemma 9.11. *For all smooth X,* $\operatorname{Gr}\mathcal{D}_X\left(X\right) \hookrightarrow \mathcal{O}_{T^*X}\left(T^*X\right)$.

Proof. $0 \to \mathcal{D}^{i-1}_X \to \mathcal{D}^i_X \to \operatorname{Sym}^i_{\mathcal{O}_X}\tau_X \to 0$, so $0 \to \Gamma\mathcal{D}^{i-1}_X \to \Gamma\mathcal{D}^i_X \to \Gamma\operatorname{Sym}^i_{\mathcal{O}_X}\tau_X$. On the other hand, $\bigoplus_i \operatorname{Sym}^i_{\mathcal{O}_X}\tau_X \simeq \mathcal{O}_{T^*X}$. ☐

Proof of the Theorem. $\varphi : \mathcal{D}^0_{n+1}/E\mathcal{D}^0_n \to \mathcal{D}_{\mathbb{P}^n}\left(\mathbb{P}^n\right)$. It's enough to show that $\operatorname{Gr}\varphi$ is an isomorphism. Now, $\mathcal{D}^0_{n+1}/E\mathcal{D}^0_n \simeq \mathcal{O}_{T^*\mathbb{P}^n}\left(T^*\mathbb{P}^n\right)$, and $\mathcal{D}_{\mathbb{P}^n}\left(\mathbb{P}^n\right)$ is embedded into $\mathcal{O}_{T^*\mathbb{P}^n}\left(T^*\mathbb{P}^n\right)$, so we get the inverse map. ☐

9.1. **Twisted differential operators on the projective space.** In Theorem 6.6 we defined the algebra of differential operators on any module over the algebra of polynomials on an affine variety. Later we showed that this definition commutes with localization by polynomials. This gives the definition of the sheaf of algebras of differential operators on a coherent sheaf over any algebraic variety X. The obtained algebra is well-behaved only if the sheaf is locally free. If the sheaf is invertible (i.e. is a line bundle), this sheaf of algebras is locally isomorphic to $\mathcal{D}_{\mathbb{P}^n}$.

Definition 9.12. A sheaf of twisted differential operators on a (smooth, separated) algebraic variety X is a sheaf of $\mathcal{O}_X$-algebras that is locally isomorphic to $\mathcal{D}_X$ (in short a TDO on X).

Let us consider the case $X = \mathbb{P}^n$. Any invertible sheaf on $\mathbb{P}^n$ is isomorphic to $\mathcal{O}(s)$ for some $s \in \mathbb{Z}$. One can define $\mathcal{O}(s)$ to correspond the construction in Theorem 9.4 to the graded module $M = \mathbb{K}[x_0, \ldots, x_n]$ with grading shifted by d. Another way to define an invertible sheaf $\mathcal{F}$ is describe the automorphism of $\mathcal{O}(U_i \cap U_j)$ given by the identifications $\mathcal{F}(U_i) \simeq \mathcal{O}_X(U_i)$ for some open affine cover $\{U_i\}$ of X on which $\mathcal{F}$ trivializes. For $\mathcal{O}(s)$ we can choose the standard cover $U_i := \{x_i \neq 0\} \cong \mathbb{A}^n$ of $\mathbb{P}^n$, on which the automorphisms are given by multiplication by $(x_i/x_j)^s$.

Let us describe $\mathcal{O}(s)$ by coordinate changes. We have to compute what happens to ∂_k when we twist it by $(x_i/x_j)^s$.

$$(x_i/x_j)^{-s} \cdot \partial_k \cdot (x_i/x_j)^s = \partial_k + s\partial_k(x_i/x_j) \cdot (x_i/x_j)^{-1}, \quad (1)$$

Since $\mathcal{D}_n$ is generated as a $\mathbb{K}[x_1, \ldots x_n]$-algebra by $\xi_1, \ldots, \xi_n$, this formula defines a sheaf of twisted differential operators on $\mathbb{P}^n$. In fact, we could put any scalar $\lambda \in \mathbb{K}$ instead of s in (1) and obtain a TDO on $\mathbb{P}^n$.

Exercise 9.13. Any TDO on $\mathbb{P}^n$ is given by the coordinate changes

$$\varphi_{ij}(\partial_k) := \partial_k + \lambda\partial_k(x_i/x_j) \cdot (x_i/x_j)^{-1}.$$

Exercise 9.14. Denote by $\mathcal{D}_{\mathbb{P}^n}(s)$ the sheaf of differential operators on $\mathcal{O}_{\mathbb{P}^n}(s)$. Then the global sections functor $\Gamma : \mathcal{M}(\mathcal{D}_{\mathbb{P}^n}(s)) \to \mathcal{M}(\Gamma(\mathcal{D}_{\mathbb{P}^n}(s)))$ is exact for $s > -n$ and faithful for $s \geq 0$.

Let us find a formula for obtaining TDOs from invertible sheaves on arbitrary (smooth, separated) varieties. Recall that a 1-form on an affine variety X is an $\mathcal{O}(X)$-module morphism $\tau_X \to \mathcal{O}(X)$. A 1-form λ is called closed if its differential $d\lambda$ vanishes. The differential can be defined as the two-form given by

$$\lambda(\xi, \eta) := \xi(\lambda(\eta) - \lambda(\xi(\eta)) - \lambda([\xi, \eta]), \quad \forall \xi, \eta \in \tau_X.$$

Exercise 9.15. Let X be affine. For a closed 1-form λ on X and $\eta \in \tau_X$ define $\varphi_\lambda(\eta) := \eta + \lambda(\eta) \in \mathcal{D}(X)$. Then φ_λ extends (uniquely) to an automorphism of $\mathcal{D}(X)$ as an $\mathcal{O}(X)$-algebra. Moreover, all automorphism of $\mathcal{D}(X)$ as an $\mathcal{O}(X)$-algebra are obtained in this way.

For non-affine X, this exercise and the Chech cohomology yield that the TDOs on X are described by $\mathrm{H}^1(X, \Omega^1_{cl})$, where Ω^1_{cl} is the sheaf of closed 1-forms on X. The group of invertible sheaves on X (a.k.a. the Picard group) is isomorphic to $\mathrm{H}^1(X, \mathcal{O}_X^\times)$, where $\mathcal{O}_X^\times$ is the sheaf of invertible regular functions on X. The logarithmic derivative gives a morphism of sheaves of abelian groups $\mathcal{O}_X^\times \to \Omega^1_{cl}$, which in turn gives a group homomorphism $\mathrm{H}^1(X, \mathcal{O}_X^\times) \to \mathrm{H}^1(X, \Omega^1_{cl})$. This homomorphism describes the correspondence between invertible sheaves and TDOs. For $X = \mathbb{P}^n$ we have $\mathrm{H}^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^\times) = \mathbb{Z}$ and $\mathrm{H}^1(\mathbb{P}^n, \Omega^1_{cl}) = \mathbb{K}$. Thus Theorem 9.15 generalizes (1).

10. The Bernstein-Kashiwara theorem on distributional solutions of holonomic modules

Let X be a smooth algebraic variety defined over $\mathbb{R}$, and let $\mathcal{S}_X^*$ denote the $\mathcal{D}_X$-module of tempered distributions on X. More precisely, for every open $U \subset X$ we take $\mathcal{S}_X^*(U) := \mathcal{S}^*(U(\mathbb{R}))$, the space of continuous functionals on the Fréchet space of Schwartz functions on $U(\mathbb{R})$. Let $\mathcal{M}_{\text{hol}}(\mathcal{D}_X)$ denote the category of holonomic $\mathcal{D}_X$-modules. Our goal in this section is to prove and use the following theorem.

Theorem 10.1 (Bernstein-Kashiwara)**.** *Let $\mathcal{F} \in \mathcal{M}_{\text{hol}}(\mathcal{D}_X)$. Then*

$$\dim \operatorname{Hom}(\mathcal{F}, \mathcal{S}_X^*) < \infty.$$

Lemma 10.2 (Exercise)**.** *Let $j : Z \subset X$ be a closed embedding of smooth affine algebraic varieties defined over $\mathbb{R}$. Then $\mathcal{S}^*(Z) \simeq j'\mathcal{S}^*(X)$.*

Corollary 10.3. *It is enough to prove Theorem 10.1 for the case when X is an affine space.*

Proof. Let $X = \bigcup_{i=1}^{r} U_i$ be an open affine cover. Then

$$\mathrm{Hom}\left(\mathcal{F}, \mathcal{S}_X^*\right) \hookrightarrow \prod_i \mathrm{Hom}\left(\mathcal{F}\left(U_i\right), \mathcal{S}^*\left(U_i\right)\right)$$

by restriction. Let $\tau_i : U_i \to \mathbb{A}^{n_i}$ be closed embeddings. Then $\mathcal{S}^*\left(U_i\right) \simeq \tau_i^! \mathcal{S}^*\left(\mathbb{A}^{n_i}\right)$. Hence by the adjunction,

$$\mathrm{Hom}\left(\mathcal{F}\left(U_i\right), \mathcal{S}^*\left(U_i\right)\right) \simeq \mathrm{Hom}\left(\mathcal{F}\left(U_i\right), \tau_i^! \mathcal{S}^*\left(\mathbb{A}^{n_i}\right)\right) \simeq$$
$$\simeq \mathrm{Hom}\left(\left(\tau_i\right)_0 \mathcal{F}\left(U_i\right), \mathcal{S}^*\left(\mathbb{R}^{n_i}\right)\right)$$

Recall that the pushforward preserves holonomicity. □

From now on let $X = V := \mathbb{R}^n$ and M be a holonomic $\mathcal{D}_n$-module.

Definition 10.4. Let ω be the standard symplectic form on $V \oplus V^*$. Denote by $p_V : V \oplus V^* \to V$ and $p_{V^*} : V \oplus V^* \to V^*$ the natural projections. Define an action of the symplectic group $\mathrm{Sp}(V \oplus V^*, \omega)$ on the algebra $\mathcal{D}(V)$ by

$$(\partial_v)^g := \pi(g)(\partial_v) := p_{V^*}(g(v,0)) + \partial_{p_V(g(v,0))}, \quad w^g := \pi(g)w := p_{V^*}(g(0,w)) + \partial_{p_V(g(0,w))}$$

where $v \in V$, $w \in V^*$, ∂_v denotes the derivative in the direction of v, and elements of V^* are viewed as linear polynomials and thus differential operators of order zero. For a $\mathcal{D}(V)$-module M and an element $g \in \mathrm{Sp}(V \oplus V^*)$, we will denote by M^g the $D(V)$-module obtained by twisting the action of $D(V)$ by $\pi(g)$.

Since the above action of $\mathrm{Sp}(V \oplus V^*)$ preserves the Bernstein filtration on $\mathcal{D}(V)$, the following lemma holds.

Lemma 10.5. *For $M \in \mathcal{M}^f(\mathcal{D}(V))$ and $g \in \mathrm{Sp}(V \oplus V^*)$ we have $\mathrm{AV}(M^g) = g\mathrm{AV}(M)$.*

Lemma 10.6. *For any $g \in \mathrm{Sp}\left(V \oplus V^*\right)$, $\mathcal{S}\left(V\right)^g \simeq \mathcal{S}\left(V\right)$, and thus $\mathcal{S}^*\left(V\right)^g \simeq \mathcal{S}^*\left(V\right)$.*

We will prove this lemma in §10.1.

Lemma 10.7. *Let $C \subset V \oplus V^*$ be a closed conic subvariety of dimension n. Then there exists a Lagrangian subspace $W \subset V \oplus V^*$, such that the projection of C onto $(V \oplus V^*)/W$ is a finite map.*

Proof. First we prove that there is a Lagrangian subspace L, such that $L \cap C = 0$. For that $\mathcal{L}$ denote the variety of Lagrangian subspaces and consider

$$Y := \left\{(\alpha, \beta) \in P\left(C\right) \times \mathcal{L} \mid \alpha \subset \beta\right\},$$

where $P\left(C\right)$ is the space of lines inside C (i.e. the projectivization). We have maps $q : Y \to P\left(C\right)$ and $q' : Y \to \mathcal{L}$, and we need to show that q' is not onto. For this it's enough to show that $\dim Y < \dim \mathcal{L}$. Now we see that

$$\dim \mathcal{L} = \frac{1}{2} n\left(n+1\right), \text{ and } \dim Y = \dim q(X) + \dim q^{-1}\left(x\right),$$

where $q^{-1}\left(x\right)$ is a generic fiber, so we calculate:

$$\dim X = \frac{1}{2} n\left(n+1\right) - 1 < \dim \mathcal{L}.$$

Now we prove the following fact: over $\mathbb{C}$, if $W \subset U$ are vector spaces, and $C \subset U$ is a conic subvariety, such that $C \cap W = 0$, then the projection $C \to U/W$ is finite. By induction on dimension we can reduce to the case $\dim W = 1$ (if it's true for $\dim W = l$ then take iterated projections, first w.r.t. W, then w.r.t. a larger subspace).

Let p be a homogeneous polynomial vanishing on C but not on W. Then

$$p(x_1, \dots, x_n) = \sum_{i=1}^{N} p_i(x_1, \dots, x_{n-1})\, x_n^i,$$

where x_i are linear coordinates, s.t. $W = \{x_1 = \cdots = x_{n-1} = 0\}$. Thus $x_n \restriction_C$ satisfies a monic polynomial over $\mathcal{O}(U/W)$. Indeed, the leading term p_N is constant — otherwise this leading term would vanish on W, so by homogeneity would have $\deg_{x_1,\dots,x_{n-1}} p_i > 0$ for all i, so p would vanish on W. Now since on $\mathcal{O}(C)$ the element x_n satisfies a monic polynomial over $\mathcal{O}(U/W)$, the ring extension $\mathcal{O}(U/W) \to \mathcal{O}(C)$ is integral, so the map $C \to U/W$ is finite.

Tanking $U := V \oplus V^*$ and $W := L$, such that the projection of C onto $V \oplus V^*/W$ is a finite map. ☐

Corollary 10.8. *For any $M \in \mathcal{M}_{hol}(\mathcal{D}_V)$ there exists $g \in \mathrm{Sp}(V \oplus V^*)$ such that M^g is smooth.*

Proof. Since M is holonomic, we have $\dim \mathrm{AV}(M) = n$. Thus, by the lemma, there exists a Lagrangian subspace $W \subset V \oplus V^*$, such that the projection of $\mathrm{AV}(M)$ onto $(V \oplus V^*)/W$ is a finite map. Since $\mathrm{Sp}(V \oplus V^*)$ acts transitively on the variety of Lagrangian subspaces, there exists $g \in \mathrm{Sp}(V \oplus V^*)$ such that $g^{-1}(W) = V^*$, and thus $g\mathrm{AV}(M)$ is finite over V. By Theorem 10.5, $\mathrm{AV}(M^g) = g(\mathrm{AV}(M))$. Thus M^g is finitely-generated over $\mathcal{O}(V)$ and thus smooth. ☐

Lemma 10.9. *Let M be a smooth $D(\mathbb{C}^n)$-module of rank r. Embed the space $An(\mathbb{C}^n)$ of analytic functions on $\mathbb{C}^n$ into $\mathcal{D}^*(\mathbb{R}^n)$ using the Lebesgue measure. Then*

$$\mathrm{Hom}(M, \mathcal{D}^*(\mathbb{R}^n)) = \mathrm{Hom}(M, An(\mathbb{C}^n)) \text{ and } \dim \mathrm{Hom}(M, \mathcal{D}^*(\mathbb{R}^n)) = \mathrm{rank}\, M,$$

where $\mathrm{rank}\, M$ *is the rank of M as a vector bundle.*

Proof. Let $M_{An} := M \otimes_{\mathcal{O}(\mathbb{C}^n)} An(\mathbb{C}^n)$ and $\mathcal{D}_{An}(\mathbb{C}^n) := \mathcal{D}_n \otimes_{\mathcal{O}(\mathbb{C}^n)} An(\mathbb{C}^n)$ be the analytizations of M and $\mathcal{D}_n$. Then

$$\mathrm{Hom}_{\mathcal{D}_n}(M, \mathcal{D}^*(\mathbb{R}^n)) \cong \mathrm{Hom}_{D_{An}(\mathbb{C}^n)}(M_{An}, \mathcal{D}^*(\mathbb{R}^n)).$$

Since M_{An} is also smooth, $M_{An} \cong An(\mathbb{C}^n)^r$. Thus it is left to prove that

$$\mathrm{Hom}_{D_{An}(\mathbb{C}^n)}(An(\mathbb{C}^n), \mathcal{D}^*(\mathbb{R}^n)) = \mathrm{Hom}_{D_{An}(\mathbb{C}^n)}(An(\mathbb{C}^n), An(\mathbb{C}^n))$$

and the latter space is one-dimensional. This follows from the fact that a distribution with vanishing partial derivatives is a multiple of the Lebesgue measure. ☐

Corollary 10.10. *If a distribution generates a smooth $\mathcal{D}$-module then the distribution is an analytic measure.*

Proof of Theorem 10.1. By Theorem 10.3 we can assume that $X = V = \mathbb{R}^n$. By Theorem 10.8 there exists $g \in \mathrm{Sp}(V \oplus V^*)$ such that $\mathcal{F}^g$ is smooth. By Theorem 10.6 we have

$$\mathrm{Hom}(M, \mathcal{S}^*(V)) \simeq \mathrm{Hom}(M^g, (\mathcal{S}^*(V))^g) \simeq \mathrm{Hom}(M^g, \mathcal{S}^*(V)).$$

Finally, $\dim \mathrm{Hom}(M^g, \mathcal{S}^*(V)) < \infty$ by Lemma 10.9. ☐

Let an algebraic group G act algebraically on a smooth algebraic variety X, both defined over $\mathbb{R}$.

Corollary 10.11. *If G has finitely many orbits on X then $\dim (\mathcal{S}^*(M))^G < \infty$.*

Proof. The Lie algebra $\mathfrak{g}$ acts on X by vector fields $\xi_\alpha, \alpha \in \mathfrak{g}$. Define a $\mathcal{D}_X$-module $\mathcal{F}$ on X by $\mathcal{F}(U) := \mathcal{D}_X(U)/\mathcal{D}_X(U)\{\xi_\alpha \restriction_U\}$. Then the solutions of this $\mathcal{D}$-module with values in $\mathcal{S}_X^*$ are exactly the G-invariant distributions. Now modulo the previous result, it remains to show that $\mathcal{F}$ is holonomic. By construction we have

$$\operatorname{Sing}\operatorname{supp} F \subset \{(x,\varphi) \in T^*M \mid \forall \alpha \in \mathfrak{g} : \langle \varphi, \xi_\alpha(x) \rangle = 0\} = \bigcup_x \mathrm{CN}_{Gx}^X$$

where CN_{Gx}^X is the conormal bundle of the orbit Gx. Since there are finitely many orbits, this is a finite union. All conormal bundles have dimension $\dim X$, so the same is true for their finite union. □

A bit more careful argument actually proves a bit stronger statement.

Theorem 10.12 (Aizenbud-Gourevitch-Minchenko)**.** *If G has finitely many orbits on X and $\mathcal{E}$ is an algebraic G-equivariant bundle on X then for any $n \in \mathbb{N}$ there is $C_n \in \mathbb{N}$, such that for any n-dimensional $\mathfrak{g}$-module τ,*

$$\dim \operatorname{Hom}_{\mathfrak{g}}(\tau, \mathcal{S}^*(X,\mathcal{E})) \le C_n.$$

Exercise 10.13. Let $\mathbb{R}$ act on $\mathbb{R}P^1$ by shifts. Compute the dimension of $(\mathcal{S}^*(\mathbb{R}P^1))^{\mathbb{R}}$.

This exercise does not use the technique of this section, but rather demonstrates the nature of the question considered in the last theorem.

10.1. Proof of Theorem 10.6.

This section requires some knowledge of representation theory.

Definition 10.14. Let $V := \mathbb{R}^n$ and let ω be the standard symplectic form on $W_n := V \oplus V^*$. The Heisenberg group H_n is the algebraic group with underlying algebraic variety $W_n \times \mathbb{R}$ with the group law given by

$$(w_1, z_1)(w_2, z_2) = (w_1 + w_2, z_1 + z_2 + 1/2\omega(w_1, w_2)).$$

Define a unitary character χ of $\mathbb{R}$ by $\chi(z) := \exp(2\pi i z)$.

Definition 10.15. The oscillator representation of H_n is given on the space $L^2(V)$ by

$$(\sigma(x,\varphi,z)f)(y) := \chi(\varphi(y) + z))f(x+y). \tag{2}$$

Note that the center of H_n is $0 \times \mathbb{R}$, and it acts on σ by the character χ, which can be trivially extended to a character of $V^* \times \mathbb{R}$.

It is easy to see that σ is the unitary induction of (the extension of) the character χ from $V^* \times \mathbb{R}$ to $H_n = (V \oplus V^*) \times \mathbb{R}$.

Lemma 10.16. *The space of smooth vectors in σ is $\mathcal{S}(V)$, and the Lie algebra of H_n acts on it by*

$$\sigma(v)f := \partial_v f,\ \sigma(\varphi)f := \varphi f,\ \sigma(z)f := 2\pi i z f. \tag{3}$$

Proof. Formula (3) is obtained from (2) by derivation. Now, it is known that the space of smooth vectors in a unitary induction consists of the smooth L^2 functions whose derivatives also lie in L^2. □

Theorem 10.17 (Stone-von-Neumann)**.** *The oscillator representation σ is the only irreducible unitary representation of H_n with central character χ.*

Idea of the proof. Let me ignore all the analytic difficulties. Consider the normal commutative subgroup $A := V \times \mathbb{R}$. Conjugation in H_n defines an action of V on the dual group of A. This action has only two orbits. The closed orbit is the singalton $\{1\}$ and the open orbit $\mathcal{O}$ is the complement to the closed one. The restriction $\sigma|_A$ decomposes to a direct integral of characters in $\mathcal{O}$, each "with multiplicity one". The restriction of any non-zero subrepresentation $\rho \subset \sigma$ to A will also include χ, and thus the whole orbit $\mathcal{O}$ of χ. Thus $\rho = \sigma$ and σ is irreducible.

Now let τ be any irreducible unitary representation of H_n with central character χ. Then the restriction of τ to A will again include all the characters in $\mathcal{O}$ with multiplicity one. Thus τ is the induction of an irreducible representation of the stabilizer of χ in H_n. However, this stabilizer is A and thus $\tau \simeq \sigma$. □

Note that the symplectic group $\mathrm{Sp}(V \oplus V^*)$ acts on H_n by automorphisms, preserving the center. Thus the theorem implies the following corollary.

Corollary 10.18. *For every $g \in \mathrm{Sp}(V \oplus V^*)$ there exists a (unique up to a scalar multiple) linear automorphism of $\mathcal{S}(V)$ such that*

Since the Lie algebra of H_n generates $\mathcal{D}_n$, this corollary implies Theorem 10.6.

Remark 10.19. The uniqueness part of Theorem 10.18 follows from Schur's lemmas. Theorem 10.18 defines a projective representation of $\mathrm{Sp}(V \oplus V^*)$ on $\mathcal{S}(V)$, i.e. a map $\tau : \mathrm{Sp}(V \oplus V^*) \to \mathrm{GL}(\mathcal{S}(V))$ such that $\tau(gh) = \lambda_{g,h}\tau(g)\tau(h)$. It is not possible to coordinate the scalars in order to obtain an honest representation of $\mathrm{Sp}(V \oplus V^*)$, but it is possible to obtain a representation of a double cover $\widetilde{\mathrm{Sp}}(V \oplus V^*)$, called the metaplectic group. This was shown by A. Weil.

11. Derived categories

Let $\mathcal{A}$ be an abelian category, and $\mathcal{C}(\mathcal{A})$ the category of complexes over $\mathcal{A}$.

Definition 11.1. Let $\varphi : C \to D$ be a morphism in $\mathcal{C}(\mathcal{A})$. We say that φ is homotopic to zero if there exists a collection of maps $\lambda_k : C_{k+1} \to D_k$ such that

$$\varphi_k = \lambda_k \circ d_k^C + d_{k-1}^D \circ \lambda_{k-1}.$$

We say that two morphisms of complexes are homotopic difference is homotopic to zero. Define the homotopy category of $\mathcal{A}$ (denoted $\mathcal{K}(\mathcal{A})$) to have complexes as objects and morphisms given by

$$\mathrm{Hom}_{\mathcal{K}}(\mathcal{A})(C, D) := \text{homotopy equivalence classes in } \mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(C, D).$$

We say that two complexes are homotopy equivalent if they are isomorphic in $\mathcal{K}(\mathcal{A})$.

The category $\mathcal{K}(\mathcal{A})$ is additive but not abelian.

Definition 11.2. A morphism $\varphi : C \to D$ in $\mathcal{C}(\mathcal{A})$ (or in $\mathcal{K}(\mathcal{A})$) is called a quasi-isomorphism if the cohomologies $H^k(\varphi)$ are isomorphisms for any k.

The derived category will be defined as the localization of $\mathcal{K}(\mathcal{A})$ by quasiisomorphisms. The idea is that this category includes slightly more information the the cohomologies of the complexes. We will also define derived functors between derived categories, and they will carry more information than the usual derived functors. In particular, we will be able to compose them, and in this way derive the composition of a left exact functor and a right exact functor.

In order to show that the derived categories are well defined we will show that the quasiisomorphisms satisfy the Ore condition. For this we will need the cone construction.

Definition 11.3. For $(C, d) \in \mathcal{C}(\mathcal{A})$ define $(Cone(C), Cone(d)) \in \mathcal{C}(\mathcal{A})$ by

$$Cone(C)_i := C_i \oplus C_{i+1}, \quad Cone(d)(a, b) := (da + b, -db).$$

Lemma 11.4. *Exercise*

(1) $Cone(C)$ is homotopy equivalent to zero.
(2) $\varphi : C \to D$ is homotopic to zero if and only if it can be extended to a morphism $\varphi' : Cone(C) \to D$.

Lemma 11.5. *Any morphism of complexes is homotopy equivalent both to an epimorphism and to a monomorphism.*

Proof. Since cones are homotopy equivalent to zero, any $\varphi : C \to D$ is homotopy equivalent to the monomorphism $\varphi' : C \to Cone(C) \oplus D$ given by $\varphi'_k(a) := (a, 0, \varphi_k(a))$ and to the epimorphism $\varphi'' : C \oplus Cone(D) \to D$ given by $\varphi''_k(a, b, c) := \varphi_k(a) + b$. □

Let us give some geometric intuition on cones. For every topological space X one can define a contractible space that includes it by $Cone(X) := X \times [0, 1]/(X \times \{1\})$. Moreover, for any continuous map $\nu : X \to Y$ we can define $Cone(\nu)$ to be the quotient of $(X \times ([0, 1]) \coprod Y)$ by the equivalence relation $(x, 0) \sim \nu(x)$. Then $Cone(\nu)$ includes Y and the quotient is the suspension $S(X) = X \times [0, 1]/(X \times \{0\} \cup X \times \{1\})$. By this analogy we will now define the cone of a morphism.

Definition 11.6. Let $\varphi : C \to D$ be a morphism in $\mathcal{C}(\mathcal{A})$. Define $Cone(\varphi) \in \mathcal{C}(\mathcal{A})$ by $Cone(\varphi) := (Cone(C) \oplus D)/\Delta C$. In other words:

$$Cone(\varphi)_i := D_i \oplus C_{i+1} \text{ with differential given by } d(a, b) = (da + \varphi(b), -db).$$

Notation 11.7. For $(C, d) \in \mathcal{C}(\mathcal{A})$ and $k \in \mathbb{Z}$, denote by $C[k]$ the complex given by

$$C[k]_i = C[k + i], \quad d[k]_i = (-1)^k d_{k+i}.$$

Lemma 11.8 (Exercise). *(1) The following short sequence of complexes is exact*

$$0 \to D \to Cone(\varphi) \to C[1] \to 0.$$

Moreover, the connecting morphism in the corresponding long exact sequence of cohomologies is $\mathrm{H}^{i+1}(\varphi)$.
(2) $Cone(D \to Cone(\varphi))$ is homotopy equivalent to $C[1]$.
(3) $Cone(Cone(\varphi) \to C[1])$ is homotopy equivalent to $D[1]$.

The triple $C, D, Cone(\varphi)$ is called an exact triangle.

Corollary 11.9. *φ is a quasi-isomorphism if and only if $Cone(\varphi)$ is an acyclic complex.*

Proposition 11.10. *The system of quasiisomorphisms in $\mathcal{K}(\mathcal{A})$ satisfies the Ore conditions. In other words for any quasi-isomorphism $\mu : C \to D$ and any morphism $q : E \to D$ there exists a quasi-isomorphism $\nu : L \to E$ and a morphism $p : L \to C$ with $\mu \circ p = q \circ \nu$.*

Proof. By Theorem 11.5 we can assume that $\mu \oplus q : C \oplus E \to D$ is an epimorphism. Let $L := \mathrm{Ker}(\mu \oplus q)$, and let $\nu : L \to E$ and $p : L \to C$ be the projections. From the short exact sequence $0 \to L \to C \oplus E \to D \to 0$ we obtain the long exact sequence

$$\cdots \to \mathrm{H}^{i-1}(D) \to \mathrm{H}^i(L) \to \mathrm{H}^i(C) \oplus \mathrm{H}^i(E) \to \mathrm{H}^i(D) \to \mathrm{H}^{i+1}(L) \to \dots$$

Since μ is a quasiisomorphism, $\mathrm{H}^i(C)$ is mapped isomorphically to $\mathrm{H}^i(D)$, which implies that the morphism $\mathrm{H}^i(L) \to \mathrm{H}^i(E)$ is onto. Since $\mathrm{H}^{i-1}(C)$ is mapped isomorphically to $\mathrm{H}^{i-1}(D)$ we obtaine that the map $\mathrm{H}^{i-1}(D) \to \mathrm{H}^i(L)$ is zero and thus the morphism $\mathrm{H}^i(L) \to \mathrm{H}^i(E)$ is an isomorphism. Thus ν is a quasiisomorphism. ☐

Definition 11.11. Let $C, D \in \mathcal{K}(A)$. A (C, D)-triple is a triple (E, ν, φ), where $\nu : E \to C$ is a quasiisomorphism and $\varphi : E \to D$ is a morphism.

We say that two (C, D)-triples (E, ν, φ) and (E', ν', φ') are linked if there exists an (E, E')-triple (L, α, β) such that both α and β are quasiisomorphisms and

$$\nu \circ \alpha = \beta \circ \nu', \quad \varphi \circ \alpha = \beta \circ \varphi'.$$

For $L \in\in \mathcal{K}(A)$, a join of a (C, D)-triple (E, ν, φ) and a (D, L)-triple (M, μ, ψ) is defined to be the (C, D)-triple $(N, \nu \circ \alpha, \psi \circ \beta)$, where (N, α, β) is an (E, M)-triple satisfying $\varphi \circ \alpha = \mu \circ \beta$. Note that the triple (N, α, β) satisfying the condition always exists by Theorem 11.10.

Lemma 11.12. *The link relation is an equivalence relation, and the equivalence class of the join of two equivalence classes of triples is well-defined, i.e. does not depend on the representatives and on the choice of the triple* (N, α, β).

This lemma follows from Theorem 11.10. We leave the deduction as a long exercise.

Definition 11.13. The derived category $D(\mathcal{A})$ is defined by $Ob(D(\mathcal{A})) = Ob(\mathcal{K}(\mathcal{A}))$ and for $C, D \in Ob(D(\mathcal{A}))$,

$$\mathrm{Hom}_{D(\mathcal{A})}(C, D) = \{\text{equivalence classes of } (C, D) - \text{triples}\}.$$

Lemma 11.14. *The derived category* $D(\mathcal{A})$ *is additive.*

Proof. Let $C, D \in Ob(D(\mathcal{A}))$, and let $\eta = (E, \nu, \varphi)$ and $\zeta = (L, \mu, \psi)$ be (C, D)-triples. Theorem 11.10 implies that there exists an (E, L)-triple (M, α, β) such that both α and β are quasiisomorphisms and $\nu \circ \alpha = \mu \circ \beta$. Then η is equivalent to $(M, \nu \circ \alpha, \varphi \circ \alpha)$ and ζ to $(M, \mu \circ \beta, \psi \circ \beta)$. We define their sum to be (the equivalence class of) $(M, \nu \circ \alpha, \varphi \circ \alpha + \psi \circ \beta)$. ☐

Note that the derived category is not abelian. Rather, it is a triangulated category. Note that we have well-defined cohomology functors $H^i : D(\mathcal{A}) \to \mathcal{A}$.

Definition 11.15. The truncation functors are defined as

$$\tau^{\leq n}(X) := \left(\cdots \to X^{n-1} \to \ker\left(X^n \to X^{n+1}\right) \to 0 \to \dots\right)$$

$$\tau^{\geq n}(X) := \left(\cdots \to 0 \to \mathrm{coker}\left(X^{n-1} \to X^n\right) \to X^{n+1} \to \dots\right)$$

Then we have natural transformations $\tau^{\leq n}(X) \to X$, $X \to \tau^{\geq n}(X)$, which are isomorphisms if X $\mathrm{H}^k(X) = 0$ for any $k > n$ (resp. $k < n$).

$\tau^{\geq n}$ (resp. $\tau^{\leq n}$) is a (co)reflection onto the subcategories of complexes bounded from above (below). For any X the morphisms $\tau^{\leq n}X \to X \to \tau^{\geq n+1}X$ form an exact triangle.

Definition 11.16. For a subset $S \subset \mathbb{Z}$ define $D^S(\mathcal{A})$ to be the subcategory of $\mathcal{D}(\mathcal{A})$ consisting of objects C with $H^k(C) = 0$ for $k \notin S$. Define $D^b(\mathcal{A}) := \bigcup_{\text{finite}S} D^S(\mathcal{A})$.

Remark 11.17. $D^b(\mathcal{A})$ is equivalent to the category of bounded complexes, with link relation through bounded complexes. We will not have time to prove that.

Lemma 11.18. $\mathcal{A} \cong D^{\{0\}}(\mathcal{A})$

Proof. The functors are given by $A \mapsto (\cdots \to 0 \to A \to 0 \to \dots)$ and $C \mapsto \mathrm{H}^0(C)$. One composition is the identity. To see that the other composition is isomorphic to identity consider the isomorphisms $C \to \tau^{\geq 0} C$ and $\mathrm{H}^0(C) \to \tau^{\geq 0} C$. ☐

We will say that an object is glued from two others if together they form an exact triangle. We will say that it is glued from some set S of objects if it is glued from two others, each of which is glued from some proper subset of S.

Exercise 11.19. Let $a \leq b \in \mathbb{Z}$ and let $I := \mathbb{Z} \cap [a, b]$. Then any $D^I(\mathcal{A})$ is glued from $D^{\{a\}}(\mathcal{A}), D^{\{a+1\}}(\mathcal{A}), \dots D^{\{b\}}(\mathcal{A})$.

Definition 11.20. A bicomplex in $\mathcal{A}$is a collection of objects $B_{ij} \in \mathcal{A}$ parameterized by $\mathbb{Z}^2$ and two collections of morphisms $d_1^{ij} : B_{ij} \to B_{i+1,j}$ and $d_2^{ij} : B_{ij} \to B_{i,j+1}$ such that $d_1^2 = 0$, $d_2^2 = 0$, and $d_1 d_2 + d_2 d_1 = 0$.

For a bicomplex $B = (B_{ij}, d_1^{ij}, d_2^{ij})$ define its total complex $(Tot(B), d)$ by

$$(Tot(B))_k := \bigoplus_{i+j \geq k} B_{ij}, \quad d = d_1 + d_2.$$

Note that we can obtain a bicomplex from a complex of complexes by changing the sign of differentials in every odd column.

Lemma 11.21 (Grothendieck)**.** *Let (B, d_1, d_2) be a bicomplex, and assume that d_1 is acyclic, and on any diagonal $i + j = k$, $B_{ij} = 0, i \gg 0$. Then its total complex* $\operatorname{Tot} B$ *is acyclic.*

Proof. Let $c \in Tot(B)_k$ with $dc = 0$. Let N be s.t. $B_{i,k-i} = 0$ for all $i > N$.

We want to show that $c = dx$ for some $x \in Tot(B)_{k-1}$. We do this by induction on l s.t. $c_{i,k-i} = 0$ for all $i < N + 1 - l$. As a base we take $l = 0$. Then $c = 0$. For the induction step, assume $c_{i,k-i} = 0$ for all $i < N + 1 - l$, and let $\alpha := c_{N+1-l,k-N-1+l}$. Then $d_1\alpha = 0$, thus $\alpha = d_1\beta$ for some $\beta \in B_{N+1-l,k-N+l}$. Then $c \sim c' := c - d\beta$, and $c'_{N+1-l,k-N-1+l} = 0$. Thus $c' = dx'$ by the induction hypothesis. Now, $c = d(\beta + x')$. ☐

Corollary 11.22. *If $\nu : B \to B'$ is an isomorphism of bicomplexes that satisfy the support condition as above. Suppose ν is a d_1-quasi-isomorphism. Then* $\operatorname{Tot} \nu$ *is a quasi-isomorphism.*

Corollary 11.23. *If B is acyclic except at row 0 and satisfies the support condition as above then* $\operatorname{Tot} B$ *is quasi-isomorphic to the cohomology complex $H^{0,\bullet}(B)$.*

Proof. Let $B_{\bullet j}$ denote the j-th column of B. Consider the exact triangle of complexes:

$$\tau^{<0} B_{\bullet j} \to B_{\bullet j} \to \tau^{\geq 0} B_{\bullet j}.$$

The first one is acyclic, and thus $B_{\bullet j} \to \tau^{\geq 0} B_{\bullet j}$ is a quasi-isomorphism. We get a d_1-quasi-isomorphism of bicomplexes $B \to \tau^{i \geq 0} B$. By the previous corollary this implies a quasi-isomorphism $Tot(B) \to Tot(\tau^{i \geq 0} B)$.

In the same way, the exact triangle
$$\tau^{<1}(\tau^{\geq 0}B_{\bullet j}) \to \tau^{\geq 0}B_{\bullet j} \to \tau^{\geq 1}B_{\bullet j}.$$
gives a quasi-isomorphism $\tau^{i<1}(\tau^{i\geq 0}B) \to \tau^{i\geq 0}B$, and by taking total complexes, a quasi-isomorphism $H^{0,\bullet}(B) \to Tot(\tau^{i\geq 0}B)$. Together, we get isomorphisms in the derived category between $Tot(B)$, $Tot(\tau^{i\geq 0}B)$ and $H^{0,\bullet}(B)$. □

Now we would like to define derived functors. Suppose that $\mathcal{A}$ has enough injective objects.

Lemma 11.24. *Any $C \in \mathcal{C}^{\geq 0}(\mathcal{A})$ has an injective resolution, i.e. is quasi-isomorphic to a complex consisting of injective objects.*

Proof. First of all, let us show that C can be embedded into an injective complex. Embed C_0 into an injective I_0, and C_1 into (an injective) I_1. Then the composed map $C_0 \to I_1$ can be lifted (by the injectivity of I_1) to $d_0 : I_0 \to I_1$. Then we embed C_2 into (an injective) I_2, and lift the map $C_1/d_0^C(C_0) \to I_2$ to a map $d_2' : I_1/d_0(I_0) \to I_2$. We continue building I by induction.

Now we embed C into an injective complex I^0, then I^0/C into I^1 and so on. In this way we construct a bicomplex $0 \to I_0 \to I_1 \to \dots$. By Theorem 11.23 the total complex will be quasi-isomorphic to C. □

Lemma 11.25 (Exercise)**.** *Let I, J be bounded on the left complexes consisting of injective objects, and let $\varphi : I \to J$ be a quasi-isomorphism. Then φ is an isomorphism in the homotopic category.*

Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor.

Definition 11.26. For any $C \in D(\mathcal{A})$ choose an injective resolution I and define $DF(C) := F(I)$. This defines a functor $DF : D(\mathcal{A}) \to D(\mathcal{B})$.

We say that an object $X \in \mathcal{A}$ is F-acyclic if $F(X) \in D^{\{0\}}\mathcal{B}$.

Proposition 11.27 (Exercise)**.** *Let C be a bounded on the left complex consisting of F-acyclic objects. Then $DF(C) \cong F(C)$.*

References

[AGM] A. Aizenbud, D. Gourevitch, A. Minchenko: *Holonomicity of relative characters and applications to multiplicity bounds for spherical pairs*, arXiv:1501.01479.

[Ber72] J. Bernstein: *Analytic continuation of generalized functions with respect to a parameter*, Functional Analysis and its Applications 6, No.4, 26-40 (1972).

[Ber] J. Bernstein, *A course on D-modules*, available at www.math.uchicago.edu/~mitya/langlands.html.

[Bor87] A. Borel: *Algebraic D-Modules*, Perspectives in Mathematics, 2, Boston , MA (1987): Academic Press, ISBN 0121177408.

[Cou95] S. C. Coutinho: *A primer of algebraic D-modules,* London Mathematical Society Student Texts **33**. Cambridge University Press, Cambridge, xii+207 pp. (1995).

[HTT08] R. Hotta, K. Takeuchi, T. Tanisaki, Toshiyuki *D-modules, perverse sheaves, and representation theory.* Progress in Mathematics, **236** xii+407 pp.. Birkhauser, Boston, MA, ISBN: 978-0-8176-4363-8 (1995,2008).

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