D-modules-Lecture-1

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D-
modules

Lectures on D-modules. Spring 2021

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1. Lecture 1. Recollection

1.1. Left and right D-modules. One category – two realizations

X - smooth variety (over $\mathbb{C}$)

$\mathcal{D}_X$ - sheaf of diff. operators on X

$$\mathcal{M}^{\ell}(\mathcal{D}_X) \longleftarrow \mathcal{M}(\mathcal{D}_X) \longrightarrow \mathcal{M}^{R}(\mathcal{D}_X)$$

$M_\ell \qquad\qquad M_r$

$$M_r = \omega \underset{\mathcal{O}_X}{\otimes} M_\ell$$

$\mathcal{M}^{\ell}(\mathcal{D}_X)$ - q.coh. sheaves of left D-modules.

1.2. **Coherent D-modules. 1.** Extension principle.

2. Many ~~projective~~ locally projective coherent D-modules

Extension principle.

$U \subset X$ open

$\mathcal{F}$ - D_X-module

$\mathcal{F}|_U \supset \mathcal{H}$, $\mathcal{H}$ is coherent D_U-module

$\exists$ $\mathcal{F}' \subset \mathcal{F}$, $\mathcal{F}'$-coherent D_X-module s.t. $\mathcal{F}'|_U = \mathcal{H}$.

$P = D_X \underset{\mathcal{O}_X}{\otimes} L$, L - loc. proj. $\mathcal{O}_X$-module. $\Leftrightarrow$ loc. free vector bundle

Claim. Let $\mathcal{F}$ be a coherent D_X-module. Then $\exists$ a coher. loc. proj. D-mod P and epimorph. $P \to \mathcal{F} \to 0$

1.3. **Derived category of D-modules. 1.** Bounded resolutions.

2. Category $D(X)$ $= D^b(X)$

3. Category $D_c(X)$

Def1. $D_c(X) = D(\mathcal{M}_{coh}(\mathcal{D}_X))$

Def2. $D_c(X) = \{ \mathcal{F}^\cdot \in D(X) \mid$
$H^i(\mathcal{F}^\cdot)$ are coherent$\}$

1.4. **Basic functors between D-module categories.**

1. Functors π_* and $\pi^!$

Composition

2. Base change

3. Projective morphism preserve coherence.

$\pi : X \to Y$

$\pi^! : D(\mathcal{D}_Y) \to D(\mathcal{D}_X)$ $\mathcal{M}^L$

$\pi_* : D(\mathcal{D}_X) \to D(\mathcal{D}_Y)$ $\mathcal{M}^R$

$\mathcal{F} \in \mathcal{M}^R(\mathcal{D}_X)$

$\mathcal{F} \longleftrightarrow \mathcal{F} \otimes_{\mathcal{D}_X} \mathcal{D}_{X \to Y} \to \pi_\bullet(\mathcal{F} \otimes \mathcal{D}_{X\to Y})$

$R\pi_\bullet \circ (\otimes)$

$\pi : X \to Y$

$\pi^\circ : \mathcal{M}^L(\mathcal{D}_Y) \to \mathcal{M}^L(\mathcal{D}_X)$

$\pi^\circ(\mathcal{M}) = \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{M}$

$\mathcal{D}_{X\to Y} := \pi^\circ(\mathcal{D}_Y)$

$\pi^!(\mathcal{M}) = L\pi^\circ(\mathcal{M})\,[\ \ ]$

1.5. **Duality.** **1.** Duality on the category $D_c(X)$

2. Toy example

$A = \mathbb{C}[x_1 \dots, x_n]$

$A = \mathcal{O}(X)$, X affine smooth var

$\mathcal{M}(A) \supset \mathcal{M}^{fg}(A) \supset F(A)$ – finite dim. A-mod.

$D : V \to V^*$

$D : F(A) \to F(A)$.

$$\mathbb{D}: D^{fg}(A) \to D^{op}(A)$$

$$\cup$$

$$\mathbb{D}(\mathcal{F}) \longrightarrow \mathbb{D}(\mathcal{F})$$

$$\mathbb{D}(\mathcal{F}^\cdot) = R\mathcal{H}om(\mathcal{F}^\cdot, \Omega_X)[\dim X]$$

$$\mathbb{D}(\mathcal{F}^\cdot) = R\mathcal{H}om(\mathcal{F}^\cdot, \mathcal{D}_X)^{r\to l}[\ ]$$

1.6. **Dual of a smooth D-module.**

$\mathcal{F}$ - $\mathcal{D}_X$-module.

$\mathcal{F}$ is called smooth (or $\mathcal{O}$-coherent) if it is coherent as $\mathcal{O}$-module.

Eg $\mathcal{F} = \mathcal{O}_X$

Claim. $\mathcal{F}$ is smooth $\Rightarrow$ $\mathcal{F}$ is loc. free as $\mathcal{O}_X$-module.

$\mathcal{F} \to$ vector bundle L with flat connection.

$\mathcal{F} \leftrightarrow (L, \nabla)$

$\mathbb{D}F \leftrightarrow (L^*, \nabla^*)$.

$\mathbb{D}\mathcal{O}_X = \mathcal{O}_X$

$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X, \mathcal{D}_X) = \omega_X [\dim X]$

X - real alg. variety

E - dist. on X

$\mathcal{M} \cdot M = \mathcal{D}_X(E) = \mathcal{D}_X / \mathrm{Ann}(E)$

M - $\mathcal{D}$ - module -

$\mathrm{Sol}(M) = \mathrm{Hom}_{\mathcal{D}_X}(M, \mathrm{Dist}(X))$

1.7. **Duality and functors. 1.** Preservation of duality by direct image of projective morphisms.

3. Shifted preservation of duality by pull back of smooth morphisms

$i : X \to Y$

$i_* : \mathcal{M}^R(\mathcal{D}_X) \to \mathcal{M}^R(\mathcal{D}_Y)$

$i_*(\mathbb{D}\mathcal{F}^\bullet) = \mathbb{D}(i_* \mathcal{F}^\bullet)$

$i_* \mathcal{M}(\mathcal{D}_X) \xrightarrow{\sim} \mathcal{M}_X(\mathcal{D}_Y)$

$p : X = Y \times \mathbb{P}^N \to Y$

!

$\pi : X \to Y$ smooth

of relative dim d.

$\mathbb{D}\pi^! \mathcal{F} = \pi^! \mathbb{D}(\mathcal{F})[\pm d]$

$i : X \to Y$ closed embed.

$i_* : \mathcal{M}(D_X) \xrightarrow{\sim} \mathcal{M}_X(D_Y)$
$i_* : D(D_X) \xrightarrow{\sim} D_X(D_Y)$
$D_X(D_Y) = \{ \mathcal{F} \in D(Y) \mid \mathcal{F}|_{Y-X} = 0 \}.$

1.8. **Singular support** $SS(M)$**. 1.** Kashiwara's lemma.

2. Lower bound on dimension of singular support

3. Holonomic modules. Finite length.

4. Smooth modules. Category $D_s(X)$.

5. Duality on holonomic modules and on smooth modules.

$i : X \to Y$

$\dim SS(i_* \mathcal{F}) = \dim SS\mathcal{F}$

Thm. M - coherent D_X-mod.
$M \neq 0 \Rightarrow \dim SS(M) \geq \dim X$

M is called holonomic if
M is coherent and
$\dim SS(M) \leq \dim X$.

$\mathcal{E}$ is smooth $(\mathcal{L}, \nabla)$
$\mathbb{D}\mathcal{E}$ is smooth $(\mathcal{L}^*, \nabla^*)$

$\mathbb{D} : D_s(X) \circlearrowleft$

$\mathbb{D} : D_h(X) \to D_h(X)$

$\mathbb{D} : \mathcal{H}ol(D_X) \to \mathcal{H}ol(D_X)$

$\mathbb{D}$:

$\mathcal{F}$ holonomic if it is coh. and $\mathbb{D}\mathcal{F}$ is a module

1.9. **Category $D_h(X)$ of holonomic complexes. 1.** Extension principle

2. Theorem. Preservation by all functors

3.. Two criteria of holonomicity

Ext. principle

$U \subset X$

$\mathcal{F}$ – $\mathcal{D}_X$ – mod.

$H \subset \mathcal{F}|U$, U-holon

can extend to holon. submodule in $\mathcal{F}$ on X.

Thm. All functors preserve holonomicity, preserve cat D_{hol}

Main step.

X – affine, $U \subset X$ open affine

$i_*: \mathcal{M}(\mathcal{D}_U) \to \mathcal{M}(\mathcal{D}_X)$

maps holon. into holonomic

Corollary 1 $j: U \to X$ open

$j_*: D_{hol} \to D_{hol}$.

Corol. 2 $i: X \to Y$ closed imbed.

$i^!: D(Y) \to D(X)$ preserves hol.

$U = Y \setminus X$

F-holonomic D_Y-modules

Cone $(F \to j_*(F|U))$ is supported on X and is essentially equal to $i^!(F)$

$D_X(D_Y) = D(D_X)$ Kashiwara

characterization of holonomic

1) $F^\cdot$ - coherent D_X-mod.

Then F is holonomic if $i_x^! F$ is f.dim. for $x \in X$

2) F is holonomic iff for any open $U \subset X$

$p_*(F|U) \in D(pt)$ is f.dim

$p: Y \times \mathbb{P}^N \to Y$ projection

$\cup$

X

$F^\cdot \in D_{hol}(X)$

$p_*(F^\cdot)$

1.10. **Classification of holonomic irreducible modules.** Modules $L(S, E)$

$\mathcal{F}$ - irred. holon. mod.
consider its support S.
$\exists$ open subset $U \subset S$
U - smooth,
$\mathcal{F}|_U$ is smooth.
$\mathcal{F} \Longrightarrow U$ - smooth subvar.
$\mathcal{E} \in \mathcal{M}_s(D_U)$ - module
$i: U \to X$, i - affine
$i_!(\mathcal{E}) \to i_*(\mathcal{E})$
$\mathcal{F} = \mathrm{Im}(i_!(\mathcal{E}) \to i_*(\mathcal{E})) =$
$i_{!*}(\mathcal{E})$

1. D-modules and D-complexes on singular varieties.

2. Holonomic modules and complexes on singular varieties

No notion of Smooth D-modules on singular varieties

X - singular.

$i : X \to W$, W - is smooth var.

$\mu(D_X)^{\cdot} := \mu_*(D_W)$

$D(D_X)^{\cdot} = D_*(D_W)$

$D(D_X) = D^b(\mu(D_X))$

$X = pt.$ $i : X \to \mathbb{A}^1$

$\mu(D_{pt}) = \mu_0(\mathbb{A}^1_{pt})$

X - alg. var. non-singular.

$\mathcal{O}_X$

$p : X \to pt$

$p_*(\mathcal{O}_X) = D_{pt}$ - complex

$Y = \mathbb{A}^1$, $X = \mathbb{A}^1 \setminus 0$

$j : X \to Y$

$\mathcal{K} = j_* : \mathcal{O}_X$

$0 \to \mathcal{O}_Y \to \mathcal{K} \to \Delta \to 0$

, $\Delta = i_*(\mathcal{O}_{pt})$

$\mathbb{D}(\mathcal{K})$

$0 \to \Delta \to \mathbb{D}(\mathcal{K}) \to \mathcal{O}_Y \to 0$

Δ $\mathcal{O}_Y$ Δ

$0 \to \Delta \to F \to \mathcal{K} \to 0$

$\mathbb{D}(F) = F$

$X \subset Y$ open.

:

X - sing. var. irred.

$p: Z \to X$ resol. of sing.

$U \subset X$ open smooth

$\exists$ unique irreducible

$\mathcal{L}$ s.t. $\mathcal{L}|_U = \mathcal{O}_U$

$\mathcal{L} = \mathcal{J}(U, \mathcal{O}_U)$

$H^*(X, \mathcal{L}) = H^*(Z, \mathcal{O}_Z)$

$Z \to X$ is small resol. of sing.