D-modules-Lecture-10

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10. Lecture 10. Ramarks on RH correspondence and its corollaries

10.1. **Remarks on the Theorem de Finitude.** Why this is called finiteness theorem.

The idea is that $D_{con}(X)$ can be considered as a collection if **finite** objects in $D(Sh(X_{top}))$.

(i) They can be described by finite amount of data.

(ii) Analogy with the category of vector spaces.

We have a Verdier duality $\mathbb{D} : D(Sh(X_{top})) \to D(Sh(X_{top}))$ and a morphism of functors $Id \to \mathbb{D} \circ \mathbb{D}$.

This morphism is an identity on the subcategory $D_{con}(X)$.

10.1.1. *Proof of the reduction claim in The Theorem de Finitude.*

Definition. Let X be an algebraic variety, $W \subset X$ its non-empty closed subvariety and $i : W \to X$ the corresponding inclusion. We say that W is a **retract** of X if sponding inclusion. We say that W is a **retract** of X if there exists a morphism $p : X \to W$ such that $p \circ i = Id_W$ (this morphism p is called a retraction morphism).

General informal idea is that locally such morphism should exist, i.e. there should exist a non-empty open subset $W_0 \subset W$ and its neighborhood $X_0 \subset X$ such that W_0 is a retract of X_0.

This is correct in analytic situation, and clearly wrong in Zariski topology. However if we pass to etale topology, then this becomes correct.

Proposition 10.1.2. *Let $W \subset X$ as before. Then there exists an open subset $U \subset X$,an unramified finite morphism $\nu : Y \to U$ and a non-empty closed subset $W_Y \subset Y$ such*

(i) the variety W_Y is a retract of Y and

(ii) the morphism ν defines an isomorphism $\nu : W_Y \to W_U = W \cap U \subset W$.

Using this proposition we can reduce study of many problems for pair (X, W) to pair (U, W_U) and then to retract pair (Y, W_Y).

Proof of the Proposition.

(i) We can assume that X and hence W are affine.

(ii) Consider a finite epimorphism $\pi : W \to V$, where V is an affine space. Extend this t a morphism $\pi : X \to V$

(iii) Passing to an open subset $V' \subset V$ (and taking its preimages) we can assume that we have a morphism $\pi : X \to V'$ that is finite and unramified on W.

(iv) Denote by Y the fibered product $Y = W \times_{V'} X$ and denote by W_Y the image of the diagonal embedding of W.

When we have a retraction morphism $p : X \to W$ we can further simplify the situation.

For example, we can replace X by $X' = X \times W$, consider X as a subvariety of X' identifying it with the graph of the retraction morphism p. Then we reduce our problem with the case when $X = W \times Z$ and p is the projection.

If Z is an affine space we can reduce to the situation to the case of the projection $p : X = W \times Z \to W$ where $i : W \to X$ is a coordinate embedding

After this we can replace the affine space Z by the corresponding projective space $\mathbb{P}$.

$i : W \to W \times Z \xrightarrow{pr} W$

$i(w) = (w, j(w))$

$a : W \times Z \circlearrowleft$

$a(w, z) = (w, z - j(w))$

If Z is a vector space,

$i' : w \mapsto (w, 0)$

We can embed $Z \subset \mathbb{P}$

$X \supset W \times \mathbb{P} \xrightarrow{pr} W$

$W \to W \times \mathbb{P}$ $\quad w \mapsto (w, 0)$

10.2. **Analytic corollaries of RH.**

10.2.1. *Solution functor.* Let X be a smooth algebraic variety over $\mathbb{C}$. Consider an arbitrary sheaf R of D_{an}-modules on the complex variety X_{an}.

Examples 10.2.2. (i) $R = O_{an}$

(ii) $R = O_{x,an}$ for some point $x \in X$.

(iii) $R = O$ for some point $x \in X$

(iii) $R = O_{x,for}$ for some point $x \in X$

(iv) Suppose X has a real structure and $R = C^\infty(X_\mathbb{R})$ – the sheaf of smooth functions on the manifold of real points of X.

(v) X has real structure and $R =$ **Gen-Func**$(X_\mathbb{R})$

Given a coherent $\mathcal{D}_X$-complex M we define a complex of sheaves Sol(M, R) on the topological space X_{top} by $Sol(M, R) := Hom_{D_{an}}(M_{an}, R)$ (morphisms in the derived category)

As a special case we set $Sol(M) := Sol(M, O_{an})$.

This functor is closely related to the functor Ω. Namely,

$$Sol(M) = \Omega(\mathbb{D}(M))[-dimX]$$

10.2.3. *Analytic properties of RS systems.*

Proposition 10.2.4. *Let M be an RS $\mathcal{D}_X$-complex.*

(i) For every point $x \in X$ we have $Sol(M, O_{x,an}) = Sol(M, O_{x,for})$

(ii) Suppose X has a real structure and $Y = X_\mathbb{R}$.

Then $Sol(M, Gen{-}Fun(Y)) = Sol(M, Hyperfunctions(Y)$

In other words, any hyperfunction solution of this system is in fact a generalized function.

(i) $x \in X$

$Sol(M, O_{x,an}) = Sol(M, O_{x,for})$

$N = \mathbb{D}(M)$

$Sol(M, O_{x,an}) = \Omega(\mathbb{D}M)_x = i_x^!(\Omega(N))$

$Sol(M, O_{x,for}) = i_x^!(\mathbb{D}(\Omega(N)))$

10.3. **Perverse sheaves and t-structures.** We have shown that the categories $D_{RS}(\mathcal{D}_X)$ and $D_{con}(X)$ are **canonically** equivalent.

Another point of view on this is that there exists some category $D(X)$ and we deal with two realizations of this category.

This realizations have rather different shape since they have different truncation structures.

$$\begin{tikzcd} & D(\mathcal{D}_X) \arrow[dl] \arrow[dr] & & D \\ D_{con}(X) & & D_{RS}(X) \\ con(X) \arrow[u, phantom, "\cup"] & & RS(X) \arrow[u, phantom, "\cup"] \end{tikzcd}$$

$$\begin{array}{ccc} & D_{con}(X) & \\ \cup & & \cup \\ con(X) & & Perv(X) \end{array}$$ $\mathbb{C}$-coefficients

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$$\begin{array}{ccc} & D_{con}(X, \mathbb{Q}) & \\ \cup & & \cup ? \\ con(X, \mathbb{Q}) & & Perv(X, \mathbb{Q}) \end{array}$$

<u>Question</u>. Describe the category $Perv(X, \mathbb{Q}) \subset D_{con}(X, \mathbb{Q})$ directly, ~~without~~ Dual.

In language of truncations

How to answer the same question in $D_{RS}(X) \supset RS(X)$

(1) First describe the subset: $D^+_{RS}(X) \subset D_{RS}(X)$, that consists of complexes with cohom. in ≥ 0 degrees.

M - RS - module then $i_x^! (M)$ at every point $x \in X$ is in D^+

Moreover for almost all points x $i_x^! (M)$ is in degrees $\geq n = \dim X$.

Claim. Let $M \in D^+_{?}(X)$

Then for $x \in X$ we have

(i) Outside of codim 1

$i_x^! (M) \in D^{\geq n} (pt)$

(ii) Outside of codim 2

$i_x^! (M) \in D^{\geq k-1} (pt)$

(iii) Outside of codim 3

$i_x^! (M) \in D^{\geq k-2} (pt)$.

$U \subset X$ For $x \in U$

$i_x^! (M) \in D^{\geq n}$ $j : U \to X$

$M \to j_* (M|_U) \to$