D-modules-Lecture-11

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11. Lecture 11. More about RS-modules

11.1. **Analytic proof of Deligne's criterion.**

Proposition 11.1.1. ***Deligne's criterion.***
Let X be a smooth irreducible projective algebraic variety of dimension n, D its closed subset, $U = X \setminus D$. Let E be a smooth $\mathcal{D}_U$-module. We would like to consider the restriction of E to curves.

Let D_0 denote the set of smooth points of D of dimension $n-1$. Suppose we know that the restriction of E to any curve that intersects D_0 is RS at the point of intersection.

Then E is RS (i.e. its restriction to any curve is RS).

Proof

Step 1. RH correspondence for U implies that there exists a smooth algebraic RS $\mathcal{D}_U$-module E' and an isomorphism of analytic $\mathcal{D}$-modules $\nu : E_{an} \to E'_{an}$.

Step 2. Consider an algebraic $\mathcal{D}$-module $H = \mathrm{Hom}_O(E, E')$ (Hom over O_U). Then we can consider ν as a flat holomorphic section of the sheaf H_{an} on U.

Let us denote by H' the direct image of H to X.

Step 3. Since both E and E' are RS along the smooth divisor D_0 the $\mathcal{D}$-module H is also RS along D_0.

This implies that the section ν extends to a **meromorphic** section of the sheaf H'_{an} on the open subset $V = U \cup D_0 \subset X$.

Step 4. Since the complement $X \setminus V \subset X$ has codimension ≥ 2 the general theorem implies that the section ν is meromorphic on the whole of X.

Step 5. Since X is projective the section ν is algebraic. Hence the morphism $\nu : E \to E'$ is algebraic.

The same considerations show that the morphism ν^{-1} is algebraic. Hence $E \approx E'$ is RS. $\diamond$

$H^* = \mathrm{Hom}_O(H, O)$.

$s \in H$ defines a morphism

$H^* \to O_U$

[illegible] H^* is generated ([illegible]) by finite number of sections

[illegible] $\{s_1, \ldots, s_k\} =: S$

(*) $S : H \hookrightarrow O_U^S$

$S(\nu)$ – merom. on U

$\Rightarrow$ meromorphic near x

$\mathcal{H}' \subset j_* \mathcal{O}_U$

11.2. **Gabber's Involutivity theorem.** Let X be a smooth algebraic variety of dimension n, M a finitely generated $\mathcal{D}_X$-module. The singular support $S = SS(M)$ is a closed conical subset of the total space T^*X of the cotangent bundle of X.

So far in the study of $\mathcal{D}$-modules we only used the dimension of this set. But in fact it has some interesting structures important for the study of $\mathcal{D}$-modules.

The space T^*X has canonical symplectic form – it is a symplectic variety. It turns out that the subset $S = SS(M)$ is always **coisotropic**.

Coisotropic here means that for any smooth point $s \in S$ the tangent space $T_s(S)$ is a coisotropic subspace of the tangent space $T_s(T^*X)$.

This of course immediately implies that if $S \neq \emptyset$ then $\dim(S) \geq n$.

We present a proof of much more general result due to O. Gabber. Let us formulate some preliminary notions.

11.2.1. *Filtered and graded algebras and modules.* Let A be a an associative algebra with 1. We consider the increasing algebra filtration (A_i) of A. Here we have either $\mathbb{Z}_+$ or $\mathbb{Z}$ filtration.

We always assume

(i) $A_i \cdot A_j \subset A_{i+j}$, $1 \in A_0$

(ii) $A = \cup A_i$

(iii) If we work with Z-filtration we assume that A is complete with respect to induced topology.

Usually we fix a filtration on an algebra A. Afterwards, given an A-module M we can consider corresponding filtrations (M_i) on M.

Associated graded object is defined by $gr(A) = \oplus A_i/A_{i-1}$, $gr(M) = \oplus M_i/M_{i-1}$.

Graded algebra, graded modules.

gr A - graded algebra
gr M - graded grA module

M M_i

$M_i = 0$ for $i \ll 0$

Filtr. M_i is bounded below

Rees algebra, Rees module and associated graded algebras and modules.

A, A_i

$R(A) \subset A[t, t^{-1}]$ - graded alg.

$\sum t^i A_i$

$R(M) = M[t, t^{-1}]$

$\sum t^i M_i$

$R(M)$ is $R(A)$ graded module.

$R(A)^1 \ni \tau = t \cdot 1$

$\tau \in R(A)^1$ is a central element.

Not 0 divisor

$\tau : R(A) \to R(A)$ is imbedd.

$R(A) / \tau R(A) = grA$

$R(M) / \tau R(M) = grM$

Def: ~~Assume~~ We say that filtered algebra (A, A_i) is almost commutative if grA is comm.

We usually assume grA is Northerean

Lemma. Let (A, A_i) be a almost comm. algebra. Then $grA = B$ is naturally a Poisson algebra

$P : B^i \cdot B^j \to B^{i+j-1}$

$a \in A^i, c \in A^j$

$[a, c] = ac - ca \in A^{i+j-1}$

$P(\bar{a}, \bar{c}) = \overline{[a, c]}$

$A = \mathrm{Diff}[x_1, \ldots, x_n, \partial_1 \ldots \partial_n]$

Filtration by order of operator.

gr $A = \mathbb{C}[\ldots]$, deg $x_i = 0$
deg $\xi_i = 1$

$\{\ \}$, $P(x_i, x_j) = 0$ $P(\xi_i, \xi_j) = 0$
$P(\xi_i, x_j) = \delta_{ij}$

$T^* X$ $X = \mathbb{A}^n$.

Similar when X is smooth
$A = D_X$

We assume that the algebra A is quasi-commutative, that means that $gr(A)$ is commutative. We also assume that $gr(A)$ is Noetherian.

Good filtration of an A-module M.

Let $Z := Spec(gr(A))$.

For any module with a good filtration M we define

(i) A singular support $SS(M) \subset Z$

(ii) A characteristic cycle $Ch(M)$ as an element in $K^+(Z)$.

Claim. *$SS(M)$ and $Ch(M)$ are well defined, i.e. do not depend on a choice of a good filtration.*

gr M is gr$A$-module

$Z = \operatorname{Spec} gr(A)$.
gr$M$ is a sheaf on Z

supp gr $M =: S \subset Z$ - closed

Def. $SS(M) := \operatorname{supp}(gr(M))$

claim $SS(M) \subset Z$ does not depend on choice of good filtration.

$S = SS(M) = \cup S_i$ - irred. compon

$gr M|_{S_i}$ is generically of finite rank r_i

$ch(M) := \sum r_i S_i$

11.2.2. *Poisson structure on Z.* .

Operation $P(a,b)$ on $gr(A)$

Claim. *Let $M = A/J$. Set $J^0 = gr(J) \subset gr(A)$.*

Then the ideal J^0 is closed under the Poisson bracket.

Theorem 11.3. *Gabber's Involutivity Theorem.*

Suppose the algebra $gr(A)$ is Noetherian over the field k of characteristic 0.

Let M be a finitely generated A-module, $S = SS(M) \subset Z = Spec(gr(A))$. Then this subset S is Poisson.

This means that the ideal $I \subset gr(A)$ is closed with respect to the Poisson bracket P.

What does this mean in the case of $\mathcal{D}$-modules.

Proof of Gaber's theorem

The proof is based on the following proposition.

Let us consider the algebra of dual numbers $D = k[\varepsilon]/\varepsilon^2$.

For any D-module L we set $\bar{L} = L/\varepsilon L$.

We have canonical morphisms $\bar{L} \to L \to \bar{L}$. This lifts to a short exact sequence iff the D-module L is free over D.

Let A be a D-algebra, that is free as D-module. Suppose that the algebra $\bar{A}$ is commutative. Then we can define the Poison bracket on the algebra $\bar{A}$.

Let $Z = Spec(\bar{A})$. For any finitely generated A-module M define its support $S(M) \subset Z$ by $S(M) = Supp(\bar{M})$.

Proposition 11.3.1. *Suppose A that A is a finitely generated D-algebra and M a finitely generated A-module.*

Assume that A and M are free as D-modules and that the algebra $\bar{A}$ is commutative.

Then the subvariety $S = S(M) \subset Z$ is coisotropic, e.i. the corresponding ideal in the algebra $\bar{A}$ is closed with respect to the Poisson bracket.