D-modules-Lecture-13

Monday, 28 June 2021 10:14

D-
modules-

13. Lecture 13. Verdier Specialization

13.1. Structure of characteristic variety of a holonomic module. Let M be a holonomic $\mathcal{D}_X$-module, $S = SS(M)$ its singular support.

Corollary of Involutivity Theorem.
S is a Lagrangian subvariety of T^*X

Lemma 13.1.1. *Let S be aclosed irreducible conic Lagrangian subvariety of T^*X.*

Then there exists a smooth locally closed subvariety $Z \subset X$ such that S is the closure of the conormal bundle $N^*Z \subset T^*X$.

Thus, starting with a holonomic $\mathcal{D}_X$-module M, we can construct several irreducible subvarieties $Z_i \subset X$ that describe its singular support.

13.2. **Other approaches to RS-modules.**

Claim. *A holonomic $\mathcal{D}_X$-module M is RS iff there exists a good filtration of M with a property that $gr(M)$ is strictly supported on $SS(M)$.*

This means, that if we denote by $I \subset O_{T^*X}$ the ideal of functions that vanish on $SS(M)$ then $I \cdot gr(M) = 0$

This lemma is correct but I do not know how to prove it elementary. I would like to describe some way how one can approach this proof.

13.2.1. *Functor of* ***nearby cycles***. There are some functors important defined on holonomic modules that can not be expressed directly in terms of six Grothendieck functors.

Let $\mathfrak{A}$ denote the standard affine line with coordinate t, $0 \in \mathfrak{A}$ - is a closed subset of $\mathbb{A}$. We denote by $i : 0 \to \mathbb{A}$ the closed imbedding, by $\mathbb{A}^*$ the punctured line $\mathbb{A} \setminus 0$ and by j the open imbedding $j : \mathbb{A}^* \to \mathbb{A}$.

Consider a holonomic module N on $|\mathbb{A}^*$ and set $M = j_*(N)$. This is a module, and it is holonomic.

Let M' denote the maximal quotient of M supported at 0. Set $\Psi(N) := i^!(R)$.

Thus we constructed a functor $\Psi : Hol(\mathbb{A}^*) \to Hol(pt)$ via $\Psi(N) := i^!(R)$.

$j_! N \xrightarrow{v} j_* M$

<u>claim.</u> $R = j_* \cancel{M} / \mathrm{Im}\, j_! \cancel{M} = \mathrm{coker}\, v$

$M \supset L$ - submodule f. dim. gener. M

$R = M / D_0(t^N L)$ N - large

claim (i) Ψ is an exact functor.
(ii) It extends to Dhol

In fact, now we can repeat this construction in more general situation. Let X be an algebraic variety and t a regular function on X. We can interpret t geometrically as a morphism $t : X \to \mathbb{A}$.

Set $X_0 = t^{-1}(0)$ and $X^* = t^{-1}(\mathbb{A}^*)$.

Then in the same way we define the functor

$\Psi : Hol(X^*) \to Hol(X_0)$.

Exact, extends to D_{hc}

13.3. **Verdier construction.** Let X be a smooth variety and $Y \subset X$ a closed smooth subvariety.

Verdier constructs a deformation of X to the variety N^*Y – the total variety of the conormal bundle to Y in X.

Namely he constructed an algebraic variety Y and a

Namely he constructed an algebraic variety X and a morphism $t : Z \to \mathbb{A}$ such that $\mathbb{G}_m$ action on Z

(i) $Z_0 = t^{-1}(0)$ is isomorphic to N_Y^*

(ii) The variety $Z^* = t^{-1}(\mathbb{A}^*)$ is canonically isomorphic to $\mathbb{A}^* \times X$ with equivariant projection $Z^* \to \cancel{X}$ $\mathbb{A}^*$

Construction.

(1) X-affine, I - ideal of Y

$A = \mathcal{O}(X)$ $\mathcal{O}(X \times \mathbb{A}^1) = \mathcal{O}[t]$

$\mathcal{O}[t] \supset B = \bigoplus_{k \geq 0} t^{-k} I^k$

$Z = \operatorname{Spec} B$

$A = B_0 \hookrightarrow B$ $Z \to X$

$Sp : \operatorname{Hol}(\mathcal{D}_X) \longrightarrow \operatorname{Hol}(N_Y^*)$

consider case

$X = \mathbb{A}$, $Y = 0$

$Sp(N) = Sp(N) = j_{*}^{*} p^*(N) \leftarrow N$

N_Y^* $\quad Z^* = X \times \mathbb{G}_m \quad X$

Specialization functor.

$Sp : D_{hol}(X) \longrightarrow D_{hol}(N_Y^*)^{\mathbb{G}_m\text{-equiv}}$

Exact functor

Exercise. $X = \mathbb{A}$, $Y = pt$

N_Y^* - dual line.

$Sp : \mathcal{O} = \mathcal{O}$

$Sp : x^{\lambda} = \xi^{-\lambda}$

$Sp_y(e^{1/x}) = 0$

M - holon. DX-module

$Y \subset X$ be a smooth irred. subvariety

(i) $Sp_Y(M) = 0$ if Y is not one of special screen for M.

(ii) If Y is a component of M then $\dim Sp_Y(M) \leq mult_Y \ldots$

Claim. M is RS iff there is an equal.