D-modules-Lecture-14

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D-
modules-

14. Lecture 14. Correction about Verdier specialization functor

14.1. **Deformation to the normal cone.** Let X be a smooth variety, $Y \subset X$ a closed smooth subvariety. Consider a normal bundle N_Y to the subvariety Y and denote by $N_Y X$ the total space of this bundle.

Informal remark . Consider Y as a subvariety in X and in $N_Y X$ (zero section). Then tubular neighborhoods of Y in these two spaces are very close.

Formally this means that there exists a deformation of the space $N_Y X$ to X.

Claim. *There exists a smooth algebraic variety Z with the action of G_m and a morphism $p : Z \to \mathbb{A} = \mathbb{A}^1$ such that*

(i) p is G_m equivariant.

(ii) $Z_0 = p^{-1}(0)$ is isomorphic to $N_Y X$ compatible with the action of G_m.

(iii) The complement $Z^* = p^{-1}(\mathbb{A}^*)$ is isomorphic to $G_m \times X$ compatible with the action of G_m.

Construction of the deformation Z.

Consider the variety $W = \mathbb{A} \times X$ and a subvariety $Y = Y \times 0 \subset W$. The variety W has natural G_m action.

We define Z' to be a blow-up of W at Y, $p : Z' \to Y$ the natural projection.

We get Z by removing from Z' closed subset that is blow-up of $0 \times X$ at Y.

14.1.1. *Nearby cycles.* Let $p : Z \to \mathbb{A}$ be a projection. denote by t the corresponding function on Z.

We define the functor $\Psi : Hol(Z^*) \to Hol(Z_0)$

Starting with a holonomic module M consider the module $M' = M \cdot t^s$ over the ring $k[[s]]$.

Then we set $\psi(M); = Cone(j_1(M') \to j_*(M')$.

This functor is also defined on the category $D_h(\mathcal{D}_X)$. It is an exact functor

It is an exact functor.

Definition. Let X be a smooth variety and $Y \subset X$ a smooth subvariety. We define the **Specialization** functor

$Sp : D_h(X) \to D_h(N_Y X$ as $Sp(M) = Psi \circ q^!(M))$
where $q : Z^* \to X$ is the projection.

$\mathbb{G}_m$ equivariant D-modules

Let $\mathbb{G}_m$ act on variety W

M - D_W-modules

Suppose $\mathbb{G}_m$ acts on W as $\mathcal{O}$-module and compat with action on D_W

$D_W \cdot M \to M$ is $\mathbb{G}_m$-invariant.

such module is called weakly $\mathbb{G}_m$ equivariant.

Let ξ the $\mathbb{G}_m$ be the standard generator.

Two actions of ξ on M

1) $\gamma(\xi)$ - derivation of the action of $\mathbb{G}_m$

2) $\xi \to$ Vectr $\subset D_W$

$\delta(\xi)$ action of vector.

def $:= \gamma(\xi) - \delta(\xi) : M \to M$ is a morph of D-modules.

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cospecialization.

$cosp : D_{hol}(X) \to D_{hol}(N^*_Y X)$.

Four : $D(N_Y X) \to D(N^*_Y X)$.

Let $V \to X$ be a vector bundle

V^* - dual bundle.

TV - total space.

Four : $D(TV) \to D(TV^*)$

V - vector space, $x_1 \dots x_n$

V^* - coord $\xi_1 \dots \xi_n$

$D(V) = k[x_i, \dots \partial_i]$

$D(V^*) = k[\xi_i, y_i]$ $\quad y_i = \partial_{\xi_i}$

consider on the line $\mathbb{A}$
a vector D-module L_ξ
corresp. to function e^ξ

$$L_\xi = \{ e \mid \partial e = e \}$$

$$m: V \times V^* \to \mathbb{A}$$

$$m^*(L_\xi) \in D(V \times V^*)$$

$$D(V) \longrightarrow D(V^*)$$

$$M \in D(V) \longrightarrow p_1^!(M) \in D(V \times V^*) \to$$

$$\to L_\xi \otimes p_1^*(M)$$

$$\longrightarrow (p_2)_*(\qquad) \in D(V^*)$$

<u>claim</u>: On weakly G_m-equiv. modules Four maps RS to RS.

$$\mathrm{cosp}: D_h(X) \longrightarrow D_h(N^*_Y X)$$

$$\mathrm{cosp}(M) = \mathrm{Four}(Sp(M))$$

<u>Suspecture.</u>
$M \in \mathrm{Hol}(X)$.
Then for any $Y \subset X$

$$\mathrm{cosp}(M) \subset \mathrm{Hol}(N^*_Y X)$$

$$\mathrm{gr}\, M \big|_{N^*_Y X} \qquad N^*_Y X \subset T^* X$$

(i) $\mathrm{vol}\ \mathrm{cosp}\, M \leq \mathrm{vol}\ \mathrm{gr}\, M\big|_{N^*_Y X}$
(ii) M is RS iff for any Y these values are equal.

Gabber's theorem

Definition. ε-algebra is an comm. algebra A with 1 and $\varepsilon \in A$ s.t. $\varepsilon^{2} -$