# LECTURE 15 IN D-MODULES II - BEILINSON-BERNSTEIN LOCALIZATION

## 1. Semi-simple Lie algebras

Let $\mathfrak{g}$ be a Lie algebra over $k$, and let $\mathcal{U}(\mathfrak{g})$ be its universal enveloping algebra, *i.e.* the associative algebra generated by $\mathfrak{g}$ with relations $xy - yx = [x, y]$. The functor $\mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ is the left adjoint functor to the forgetful functor from associative algebras to Lie algebras. $\mathcal{U}(\mathfrak{g})$ is a quotient of the free algebra $T(\mathfrak{g}) = \bigoplus_i \mathfrak{g}^{\otimes i}$, and the natural grading on $T(\mathfrak{g})$ defines a filtration on $\mathcal{U}(\mathfrak{g})$ called the Poincare-Birkhoff-Witt filtration. Let $S(\mathfrak{g}) = \mathcal{O}(\mathfrak{g}^*)$ denote the symmetric algebra of $\mathfrak{g}$.

**Theorem 1** (Poincare-Birkhoff-Witt)**.** *The natural map* $S(\mathfrak{g}) \to \mathrm{Gr}\,\mathcal{U}(\mathfrak{g})$ *is an isomorphism.*

A Lie algebra $\mathfrak{g}$ is called *solvable* if the sequence $\mathfrak{g}_0 = \mathfrak{g}, \mathfrak{g}_n := [\mathfrak{g}_{n-1}, \mathfrak{g}_{n-1}]$ is eventually 0. $\mathfrak{g}$ is called *simple* if it has no ideals, *i.e.* is simple as a module over itself.

All the simple non-trivial algebras are classified:

$$\mathfrak{sl}_n, \mathfrak{so}_n, \mathfrak{sp}_n, E_6, E_7, E_8, F_4, G_2$$

On $\mathfrak{g} = \mathfrak{gl}_n$ we have the natural trace form: $\langle X, Y\rangle = \mathrm{tr}(XY)$. It is symmetric and $\mathfrak{g}$-invariant: $\langle [Z, X], Y\rangle = -\langle X, [Z, Y]\rangle$. Any $\mathfrak{g}$ maps into $\mathfrak{gl}_n$ using the adjoint representation. This defines the Killing form on $\mathfrak{g}$: $\langle X, Y\rangle := \mathrm{tr}(ad(X) \circ ad(Y))$.

**Definition 2.** $\mathfrak{g}$ is called semi-simple if it satisfies one of the equivalent conditions:

- (i) $\mathfrak{g}$ has no solvable ideals
- (ii) The Killing form is non-degenerate
- (iii) $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ with $\mathfrak{g}_i$ simple and non-trivial.

From now on we assume that $\mathfrak{g}$ is semi-simple, and $G$ is some connected algebraic group with Lie algebra $\mathfrak{g}$. For example, $\mathfrak{g} = \mathfrak{sl}_n$, $G = \mathrm{SL}_n$.

**Definition 3.** A Cartan subalgebra is a maximal commutative subalgebra $\mathfrak{h}$ such that $ad(x)$ is diagonalizable for any $x \in \mathfrak{h}$.

**Definition 4.** Borel subalgebra = a maximal solvable subalgebra. Denoted $\mathfrak{b}$.

Example: $\mathfrak{g} = \mathfrak{sl}_n$, $\mathfrak{h}$ = diagonal matrices, $\mathfrak{b}$=upper-triangular matrices. These choices are non-unique, but all Cartan subalgebras are conjugate, and so are all Borel subalgebras. The group corresponding to a Borel subalgebra is called a Borel subgroup. It is self-normalizing.

**Definition 5.** Flag variety $X := \mathcal{B} = G/B :=$ the variety of all Borel subalgebras.

Example: for $SL_2$ it is $\mathbb{P}^1$. For $\mathfrak{sl}_n$, it is the variety of all flags. A flag is a sequence of vector spaces $V_0 \subset V_1 \subset \ldots \subset V_n$ with $\dim V_i = i$. The flag variety is always projective.

Let $T \subset G$ be the connected algebraic group corr. to $\mathfrak{h}$, and $N_T \subset G$ be its normalizer.

**Proposition 6.** $W := N_T/T$ *is a finite group.*

$W$ is called the Weyl group. For $\mathrm{GL}_n$ and $\mathrm{SL}_n$ it is permutations.

**Example 7.** $\mathfrak{g} = \mathfrak{sl}_2$. Spanned by $\{e, h, f\}$ where

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad [h, e] = 2e \quad [h, f] = -2f \quad [e, f] = h$$

Cartan: $\mathfrak{h} = \mathrm{Span}\{h\}$, Borel: $\mathfrak{b} = \mathrm{Span}\{e, h\}$.

**Exercise 8.** Let $V$ be a finite-dimensional irreducible representation of $\mathfrak{sl}_2$.

(i) $V$ contains a highest-weight vector, i.e. a non-zero vector $v$ with $ev = 0$ and $hv = \lambda v$.
(ii) $\lambda = \dim V - 1$
(iii) $V$ is spanned by $v, fv, \ldots, f^\lambda v$.
(iv) Describe the action of $e$ and of $h$ in this basis.
(v) Conclude that for every integer $\lambda \geq 0$ there exists a unique irreducible representation $V_\lambda$ of highest weight $\lambda$.

Explicitly: $V_\lambda =$ homogeneous polynomials in 2 variables of degree $\lambda$. $e$ acts by $x\partial_y$, $f$ by $y\partial_x$ and $h$ by $x\partial_x - y\partial_y$. Hint: use the Casimir operator:

$$\Delta = h^2 + 2h + 4fe \in \mathfrak{z} := \mathfrak{z}(\mathcal{U}(\mathfrak{g})).$$

**Exercise 9.** For every (semi-simple) $\mathfrak{g}$, we have $\mathfrak{z} := \mathfrak{z}(\mathcal{U}(\mathfrak{g})) \cong S(\mathfrak{g})^{\mathfrak{g}} = \mathcal{O}(\mathfrak{g}^*)^{\mathfrak{g}}$.

**Notation 10.** Let $\chi_0 : \mathfrak{z} \to k$ denote the evaluation at $0 \in \mathfrak{g}^*$, and let $\mathfrak{z}_+$ denote the kernel of $\chi_0$. Let $\mathcal{M}(\mathfrak{g}, \chi_0)$ denote the category of $\mathfrak{g}$-modules with central character $\chi_0$, and $\mathcal{M}_f(\mathfrak{g}, \chi_0)$ denote the subcategory of finitely-generated $\mathfrak{g}$-modules.

## 2. Beilinson-Bernstein localization - formulation

Let $G$ act on $X$. This gives $\mathfrak{g} \to \Gamma(X, \mathcal{D}_X)$, and thus $\mathcal{U}(\mathfrak{g}) \to \Gamma(X, \mathcal{D}_X)$. This defines functors $Loc(M) : \mathcal{M}(\mathfrak{g}) \leftrightarrow \mathcal{M}(\mathcal{D}_X) : \Gamma$ by $\Gamma(\mathcal{F}) := \mathcal{F}(X)$ and $Loc(M) := \mathcal{D}_X \otimes_{\mathcal{U}(\mathfrak{g})} M$.

**Exercise 11.** *Loc* is left adjoint to $\Gamma$.

From now on we let $X$ be the flag variety. In this case we know that $\Gamma$ and *Loc* are equivalences of categories. Beilinson and Bernstein proved this in general. In more detail:

**Theorem 12.**

$$0 \to \mathcal{U}(\mathfrak{g})\mathfrak{z}_+ \to \mathcal{U}(\mathfrak{g}) \to \Gamma(X, \mathcal{D}_X) \to 0$$

**Theorem 13.** $\forall \mathcal{F} \in \mathcal{M}(\mathcal{D}_X)$, $\forall i > 0$, $\mathsf{H}^i(X, \mathcal{F}) = 0$ *and* $Loc(\mathcal{F}) \twoheadrightarrow \mathcal{F}$.

For $G = \mathrm{SL}_2$ we already know that both theorems hold.

**Corollary 14.** *$\Gamma$ and Loc induce equivalence of categories $\mathcal{M}(\mathcal{D}_X) \to \mathcal{M}(\mathfrak{g}, \chi_0)$, that maps coherent modules to finitely-generated ones.*

*Proof.* Step 1 The canonical map $M \to \Gamma(Loc(M))$ is an isomorphism $\forall M \in \mathcal{M}(\mathfrak{g}, \chi_0)$. $\mathcal{M}(\mathfrak{g}, \chi_0)$ is the category of modules over the algebra $\mathcal{U}(\mathfrak{g})/\mathfrak{z}_+\mathcal{U}(\mathfrak{g})$, which by Theorem 12 is isomorphic to $\Gamma(X, \mathcal{D}_X)$. Thus we have $\Gamma(X, \mathcal{D}_X)^I \twoheadrightarrow M$, and thus

$$\Gamma(X, \mathcal{D}_X)^J \to \Gamma(X, \mathcal{D}_X)^I \to M \to 0$$

Since Loc is right-exact, and by Theorem 13 $\Gamma$ is exact, we have

$$\begin{array}{ccccccc}\Gamma(X,\mathcal{D}_X)^J & \longrightarrow & \Gamma(X,\mathcal{D}_X)^I & \longrightarrow & M & \longrightarrow & 0\\ \| & & \| & & \downarrow & & \\ \Gamma(X,\mathcal{D}_X)^J & \longrightarrow & \Gamma(X,\mathcal{D}_X)^I & \longrightarrow & \Gamma(Loc(M)) & \longrightarrow & 0\end{array}$$

Step 2 The canonical map $\mathcal{F} \to Loc(\Gamma(\mathcal{F}))$ is an isomorphism $\forall \mathcal{F} \in \mathcal{M}(\mathcal{D}_X)$.
By Theorem 13, the map is onto. Let $\mathcal{K}$ denote its kernel:

$$0 \to \mathcal{K} \to \mathcal{F} \to Loc(\Gamma(\mathcal{F})) \to 0$$

By the exactness of $\Gamma$ we have

$$0 \to \Gamma(\mathcal{K}) \to \Gamma(\mathcal{F}) \to \Gamma(Loc(\Gamma(\mathcal{F}))) \to 0,$$

and by Step 1 the last map is an isomorphism. Thus $\Gamma(\mathcal{K})$=0. By Theorem 13 this implies $\mathcal{K} = 0$.

Step 3 Coherent $\leftrightarrow$ finitely generated. Exercise.

□

## 3. Applications to Harish-Chandra modules

Let $K\subset G$ be a symmetric subgroup, *i.e.* $K = G^\theta$, where $\theta$ is an automorphism of $G$ with $\theta^2 = \mathrm{Id}$. In the applications, $G$ and $K$ are defined over $\mathbb{R}$, and $K(\mathbb{R})$ is a maximal compact subgroup in $G(\mathbb{R})$.

**Definition 15.** A $(\mathfrak{g}, K)$-module is a $\mathfrak{g}$-module with a compatible action of $K$. Compatible means that the two actions coincide on $\mathfrak{k} = Lie(K)$, and that

$$\forall k \in K \text{ and } \forall \alpha \in \mathfrak{g} \text{ we have } \tau(k)\pi(\alpha)\tau(k^{-1}) = \pi(Ad(k)\alpha).$$

Denote the category of all finitely generated $(\mathfrak{g}, K)$-modules with central character $\chi_0$ by $\mathcal{M}(\mathfrak{g}, K, \chi_0)$. Harish-Chandra proved that they are all $K$-admissible, *i.e.* that the multiplicities of all irreducible representations of $K$ are finite (possibly zero). For $G = SL_n$, we have $K = SO_n$. Example: $G = SL_2$, $M = Span_{\mathbb{C}}(\{z^{2n}\})$, as complex valued even functions on the unit circle. Then

$$ez^{2n} = \frac{in}{2}z^{2n-2}+inz^{2n}+\frac{in}{2}z^{2n+2},\ hz^{2n} = nz^{2n-2}-nz^{2n+2},\ fz^{2n} = \frac{in}{2}z^{2n-2}-inz^{2n}+\frac{in}{2}z^{2n+2}$$

The category $\mathcal{M}(\mathfrak{g}, K, \chi_0)$ is equivalent to the category of certain smooth representations of $G(\mathbb{R})$ (proved by Casselman-Wallach). In the example above, $M$ corresponds to $C^\infty(\mathbb{P}^1)$. In general, the equivalence sends a smooth representation $\pi$ to the space of its $K(\mathbb{R})$-finite vectors. Thus, modules in $\mathcal{M}(\mathfrak{g}, K, \chi_0)$ (Harish-Chandra modules) are algebraic skeletons of smooth representations.

Under the Beilinson-Bernstein localization they correspond to $K$-equivariant $\mathcal{D}_X$-modules.

**Definition 16.** A $K$-equivariant $\mathcal{D}_X$-module is a $\mathcal{D}_X$-module $\mathcal{F}$ with an isomorphism $p_2^0\mathcal{F} \cong a^0\mathcal{F}$ satisfying the cocycle condition. Here, $a, p_2 : K \times X \to X$ are the action map and the projection. $\mathcal{M}(\mathcal{D}_X, K)$:=the category of all $K$-equivariant modules.

A *weakly equivariant* $\mathcal{D}_X$-module is a $\mathcal{D}_X$-module $\mathcal{F}$ with an action of $k$ s.t.

$$\forall U \subset X, \xi \in \mathcal{F}(U), d \in \mathcal{D}_X(U), k \in K \text{ we have } d^k(\pi(k)\xi) = \pi(k)(d\xi).$$

On such a module we have 2 actions of $\mathfrak{k} = Lie(K)$-one by deriving the action of $K$, and another one from $\mathfrak{k} \to \mathcal{D}_X(X)$.

**Exercise 17.** (i) A weakly -equivariant module $\mathcal{F}$ is equivariant if and only if the two actions of $\mathfrak{k}$ coincide.
(ii) $\mathcal{D}_X$ is weakly equivariant, while $\mathcal{O}_X$ is equivariant.
(iii) $\forall \mathcal{F} \in \mathcal{M}_{coh}(\mathcal{D}_X, K), SingSupp(\mathcal{F}) \subset \bigcup_{x \in X} CN^X_{Kx}$
(iv) If $K$ has finitely many orbits on $X$ then any $\mathcal{F} \in \mathcal{M}_{coh}(\mathcal{D}_X, K)$ is holonomic, and irreducible modules correspond to irreducible representation of component groups of stabilizers of points:
$\rho \in Irr(K_x/K_x^0) \to$ local system on $Kx \to \mathcal{D}_{Kx}-$module $\to ! *$-extension.

**Proposition 18** (Classical)**.** *$K = G^\theta$ has finitely many orbits on the flag variety $X$.*

**Exercise 19.** Prove this for $G = \mathrm{SL}_n$.

**Exercise 20.** The functors $Loc$ and $\Gamma$ give $\mathcal{M}_f(\mathfrak{g}, K, \chi_0) \cong \mathcal{M}_{coh}(\mathcal{D}_X, K)$

**Corollary 21.** *(i) Any $M \in \mathcal{M}_f(\mathfrak{g}, K, \chi_0)$ has finite length.*
*(ii) $\mathcal{M}_f(\mathfrak{g}, K, \chi_0)$ has finitely many classes of irreducible objects.*
*(iii) Classification of irreducibles in $\mathcal{M}_f(\mathfrak{g}, K, \chi_0)$ - atlas program.*

**Example 22.** Trivial $\mathfrak{g}$-module corresponds to $\mathcal{O}_X$ and to the open orbit $U$ on $X$. It lies in $(j_U)_*\mathcal{O}_U$, which is the principal series representation. It consists of $K$-finite smooth functions on $X$.

Casselman proved that any irreducible $M \in \mathcal{M}_f(\mathfrak{g}, K, \chi_0)$ can be embedded into the principal series. He used asymptotics of matrix coefficients. Beilinson and Bernstein can deduce the Casselman embedding theorem from their localization theorem above. As an intermediate step they prove that all the modules in $\mathcal{M}_{coh}(\mathcal{D}_X, K)$ are regular singular. This is proved orbitwise, using the fact that $K$ is reductive.

3.1. **Other central characters.** For any central character $\chi : \mathfrak{z} \to k$, we can define $\mathcal{D}^\chi_X$-the algebra of twisted differential operators (TDO), as we have done for the projective space. For $\chi$ in some lattice these are differential operators on a certain invertible sheaf $\mathcal{O}_\chi$ on $X$.

**Theorem 23** (Beilinson-Bernstein)**.**
*(i) $\Gamma : \mathcal{M}(\mathcal{D}^\chi_X) \to \mathcal{M}(\mathfrak{g}, \chi)$ is exact and essentially surjective.*
*(ii) If $\chi$ is regular then $\Gamma$ is an equivalence of categories.*

## 4. Center, nilpotent cone, and Springer resolution

Let $\mathcal{N} \subset \mathfrak{g}^* \cong \mathfrak{g}$ denote the nilpotent cone.

**Theorem 24** (Kostant)**.** *(i) $\mathcal{N} = zeroes(\mathfrak{z}_+ S(\mathfrak{g}))$*
*(ii) $\mathcal{N}$ is normal.*
*(iii) $G$ has finitely many orbits on $\mathcal{N}$.*

For $\mathrm{SL}_n$: (i) follows from the fact that $\mathfrak{z}_+$ is generated by the coefficients of the characteristic polynomial (viewed as polynomials in the matrix entries), and (iii) from Jordan's theorem. Part (ii) follows for any $G$ from the fact that orbits have even dimension. This in turn holds since they are symplectic manifolds.

To resolve the singularities of $\mathcal{N}$ we will need the momentum map $m : T^*X \to \mathfrak{g}^*$. To define it, let $x \in X$. Then the action gives $a_x : G \to X$, and $da_x : \mathfrak{g} \to T_xX$. Now,

$$m(x,\xi)(\alpha) := \xi(da_x(\alpha))$$

Explicitly, the stabilizer of $x$ is a Borel subgroup $B \subset G$ with Lie algebra $\mathfrak{b} \subset \mathfrak{g}$, and

$$m_x : T^*_xX \cong (\mathfrak{g}/\mathfrak{b})^* \cong \mathfrak{b}^\perp \subset \mathcal{N}(\mathfrak{g}^*)$$

**Theorem 25** (Springer). *$m$ is a resolution of singularities.*

*Sketch of proof.* On an open set we have an isomorphism, since regular nilpotent matrices belong to only one Borel subalgebra (preserve only one flag). The map is proper since $X$ is complete. □

## 5. Proof of Theorem 12

$\Phi : \mathcal{U}(\mathfrak{g}) \to \Gamma(\mathcal{D}_X)$, $\mathrm{gr}\,\Phi : S(\mathfrak{g}) \to \Gamma(\mathrm{gr}\,\mathcal{D}_X)$.

**Proposition 26.** $0 \to \mathfrak{z}_+ S(\mathfrak{g}) \to S(\mathfrak{g}) \to \Gamma(\mathrm{gr}\,\mathcal{D}_X) \to 0$

*Proof.* $S(\mathfrak{g}) = \mathcal{O}(\mathfrak{g}^*)$, $\mathrm{gr}\,\mathcal{D}_X \cong \mathcal{O}(T^*X)$. $\mathrm{gr}\,\Phi$ is given by $m : T^*X \to \mathfrak{g}^*$. We have

$$m = m' \circ i : T^*X \xrightarrow{m'} \mathcal{N} \overset{i}{\subset} \mathfrak{g}^*,$$

where $m'$ is a resolution of singularities, $i$ is a closed embedding, and $\mathcal{N}$ is normal. Thus $m'^* : \mathcal{O}(\mathcal{N}) \cong \mathcal{O}(T^*X)$ is an isomorphism, and $i^* : \mathcal{O}(\mathfrak{g}^*) \twoheadrightarrow \mathcal{O}(\mathcal{N})$ with kernel $\mathfrak{z}_+\mathcal{O}(\mathfrak{g}^*)$. □

**Proposition 27.** $\forall z \in \mathfrak{z}, \Phi(z) = \chi_0(z)\,\mathrm{Id}$.

*Proof.*
Step 1 $\Phi(z) \in \Gamma(X, \mathcal{D}_X)^G$

Step 2 $\Gamma(X, \mathcal{D}_X)^G = k\,\mathrm{Id}$
$\mathcal{O}(\mathcal{N})^G = k$, since there is a unique open orbit. $m'^* : \mathcal{O}(\mathcal{N})^G \to \mathcal{O}(T^*X)^G$. Since $G$ preserves the filtration on $\mathcal{D}_X$, we obtain by induction that

$$\Gamma(X, F_i\mathcal{D}_X)^G = k \ \forall i$$

Step 3 $\forall z \mathfrak{z}_+$, $z1_X = 0$.

□

*Proof of Theorem 12.* For any $p \geq 0$ define

$$I_p := \mathfrak{z}_+ \cap \mathcal{U}_p(\mathfrak{g}), \quad J_p := \mathfrak{z}_+\mathcal{U} \cap \mathcal{U}_p(\mathfrak{g}) = \sum_{i+j=p} \mathcal{U}_i(\mathfrak{g}) I_j$$

It is enough to prove exactness of the complex

$$J_p \to \mathcal{U}_p(\mathfrak{g}) \to \Gamma((\mathcal{D}_x)_p) \to 0$$

for any $p \geq 0$. We prove by induction. The base $p = 0$ is easy: $\mathcal{U}_0(\mathfrak{g}) = k = \Gamma(\mathcal{O}_X) = \Gamma((\mathcal{D}_X)_0)$. Also, $I_0 = J_0 = 0$. For the induction step let $p > 0$ and define

$$\mathcal{K}_p := S(\mathfrak{g})\mathfrak{z}_+ \cap S_p(\mathfrak{g}) = \bigoplus_{j>0} S(\mathfrak{g})_{p-j} S(\mathfrak{g})^{\mathfrak{g}}_j$$

We have the following commutative diagram

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
J_{p-1} & \longrightarrow & \mathcal{U}_{p-1}(\mathfrak{g}) & \longrightarrow & \Gamma((\mathcal{D}_X)_{p-1}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
J_p & \longrightarrow & \mathcal{U}_p(\mathfrak{g}) & \longrightarrow & \Gamma((\mathcal{D}_X)_p) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{K}_p & \longrightarrow & S_p(\mathfrak{g}) & \longrightarrow & \Gamma(\mathrm{gr}(\mathcal{D}_X)_p) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
0 & & 0 & & & &
\end{array}$$

All the rows and columns are complexes. We need to show that the second row is exact. The 1st row is exact by the induction hypothesis, and the 3rd row by Proposition 26. Thus, The 2nd column is exact by the PBW theorem, and the 3rd by left exactness of $\Gamma$. Thus, it is enough to show that the first column is exact, *i.e.* $J_p \twoheadrightarrow \mathcal{K}_p$. For this it is enough to show that $I_j \twoheadrightarrow S(\mathfrak{g})^{\mathfrak{g}}_j$ for any $j > 0$. Since $\mathcal{U}_j(\mathfrak{g}) \twoheadrightarrow S_j(\mathfrak{g})$ is a finite-dimensional map of $\mathfrak{g}$-modules, we have $\mathfrak{z} \cap \mathcal{U}_j(\mathfrak{g}) = \mathcal{U}_j(\mathfrak{g})^{\mathfrak{g}} \twoheadrightarrow S_j(\mathfrak{g})^{\mathfrak{g}}$. Thus, any $a \in S_j(\mathfrak{g})^{\mathfrak{g}}$ is the symbol of some $z' \in \mathfrak{z} \cap \mathcal{U}_j(\mathfrak{g})$. Set $z := z' - \chi_0(z') \in \mathfrak{z}_+ \cap \mathcal{U}_j(\mathfrak{g}) = I_j$. $\square$