D-modules-Lecture2

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D-
modules-

2. Regular singular D-modules

We have seen that the the category $M(D_X)$ of D-modules contains a very important subcategory $Hol(\mathcal{D}_X)$ of holonomic D-modules. This subcategory has many wonderful properties. In particular, on this subcategory (more precisely on its derived version) we can define six Grothendieck functors.

It turns out that the subcategory $Hol(\mathcal{D}_X)$ contains another important subcategory $RS(\mathcal{D}_X)$ - subcategory of **regular singular** D-modules.

This category is important in analytic applications since distribution solutions of RS D-modules have very good analytic behavior. The category RS is also preserved by all functors.

2.1. **Regular singularity in dimension 1.** Consider the case when X is a smooth curve. Let F be a holonomic D-module on X. After removing several points we can assume that F is smooth.

Let $\bar{X}$ be the smooth closure of X. Fix a point $a \in \bar{X}$. We would like to analyze the structure of F near point a.

First consider analytic picture. Choose a parameter t at point a and identify some analytic neighborhood of a with unit disc D. We set $D^* = D \setminus a$.

We get the following analytic picture. D –open unit disc, E-trivial vector bundle on D^*. A section of E we consider as a holomorphic vector function on D^* (or some open subset W of D^*).

We would like to describe sections satisfying the following ODE

(Ξ) $\quad \partial_t f = B(t) \cdot f,$

where $B(t)$ is a meromorphic matrix valued function on D holomorphic on D^*.

(i) Local existence.

(ii) V is the fiber of E at some point $b \in D^*$. Monodromy operator $\mu : V \to V$.

Remark. Classical theory about an operator of order n

2.1.1. *Fuchsian singularity.* System of equations Ξ is called **Fuchsian** if the matrix $B(t)$ has pole of order ≤ 1

Let $d := t\partial_t$. Then we can write our Fuchsian equation

Ξ as $df = Af$ where A is a holomorphic function on D.

Examples 2.1.2. (i) Suppose E is one dimensional and A is a constant function λ.

The solution is given by $f = t^\lambda$.

(ii) Suppose that $A(t)$ is the constant matrix function. We can assume that A is a Jordan matrix. The solution is explicitly written. It has entries $t^\lambda \cdot (\ln t)^k$

We will prove the following

Proposition 2.1.3. *Consider a Fuchsian system*

$df = A(t) \cdot f.$

Then it is meromorphically equivalent to a Fuchsian system with constant matrix A'

Remark. It is not correct that this equivalence can be made holomorphic at 0.

A solution f of the ODE Ξ is a multivalued holomorphic function. We can consider it as a function of t and $log(t)$. In case of Fuchsian system this function has tempered grows, i.e. it satisfies inequality of the type $|f(t)| \leq C|t|^{-n} \cdot (\ln t)^k$.

Definition. A system of equations Ξ is called **Regular singular** or **RS** if all its solutions have tempered growth.

We will see that any RS equation Ξ is meromorphically equivalent to a Fuchsian equation.

Remark. If we have Fuchsian system of dim ≥ 1 then we can conjugate it by meromorphic matrix function such that it will be not Fuchsian (but of course RS).

2.1.4. *Algebraic definition of Regular singularities for curves.*

We would like to give a purely algebraic definition of this notion.

Let X be a smooth curve, $s \in X$. Set $U = X \setminus a$. We denote by $j : U \to X$ the open imbedding.

Choose a local parameter t at point s and consider the vector field $d = t\partial_t$ on X. Denote by $\mathcal{D}'_X$ the sheaf of subalgebras in $\mathcal{D}_X$ generated by $\mathcal{O}_X$ and the operator d.

Let E be a smooth D-module on U. Consider $\mathcal{D}_X$-module $F = j_*(E)$.

Definition. We say that E is **RS** at point s if F has an $\mathcal{D}'_X$-submodule E' that is O-coherent and its restriction to X^* equals E. Such a module E' we will call a **lattice** in F.

We say that E is RS if it is RS at all points s in the smooth completion of U

It is clear that if E' is a lattice , then for any $k \in \mathbb{Z}$ the sheaf $t^k E'$ is also a lattice. This implies that if E is RS then any O-coherent submodule of F lies inside some lattice in F.

2.1.5. *RS D-modules in dimension 1.*

Definition. Let X be a curve, F a $\mathcal{D}_X$-module. We say that F is RS if it is holonomic and its restriction E to an open dense subset U is RS at all points s of the smooth completion Y of the curve U.

A complex F of $\mathcal{D}_X$-modules is called RS if it is holonomic and all its cohomology modules are RS.

Proposition 2.1.6. *In dimensions* 0 *and* 1 *the categories* D_{RS} *are preserved by all functors* $\mathbb{D}, \pi_*, \pi^!, \pi_!, \pi^*$.

2.1.7. *Regular singularity along smooth divisor.*

Consider now more general situation.

Let X be a smooth algebraic variety of dimension n, $S \subset X$ a closed smooth divisor, $U = X \setminus S$, $j : U \to X$ the open imbedding.

We denote by $\mathcal{D}'_X$ the sheaf of subalgebras of $\mathcal{D}_X$ generated by $\mathcal{O}_X$ and by vector fields tangent to S

Let E be a smooth $\mathcal{D}_U$-module , $F := j_*(E)$. We call an S-lattice in F an coherent $\mathcal{O}_X$-submodule E' such that

(i) Restriction of E' to U coincides with E

(ii) E' is $\mathcal{D}'_X$ -invariant.

Definition. We say that E is RS with respect to S if the sheaf F has a lattice.

In the same way as before this implies that any O-coherent subsheaf of F lies inside some lattice.

To 2.1.6 $\pi : X \to Y$, X, Y - curves
π dominant.
$\mathcal{M}$ - holon. $\mathcal{D}_Y$ - module.
Then $\mathcal{M}$ is RS iff
$\pi^! \mathcal{M}$ is RS

2.1.8. *Testing smooth D-modules on curves.* Let us call a test curve the morphism $\nu : (C, s) \to (X, S)$, where C is a smooth curve, $s \in C$, $\nu : C \to X$ a morphism such that $nu(s) \in S$ and $nu(C \setminus s) \subset U$.

We will prove the following key criterion of RS

Proposition 2.1.9. *E is RS with respect to S iff for any test curve $\nu : (C, s) \to (X, S)$ the D-module $\nu*(E)$ is RS at the point s.*

corollary. Let E be a smooth D_U-module. Suppose S is irreducible and E is RS near some point $s \in S$. Then it is RS on the whole of S.

Precisely we have open $W \subset X$, $W \ni s$, $E|_W$ is RS at $S \cap W$. Then E is RS wrto S.

Proof consider $E'' \subset F|_W$ a lattice. Then E'' is $\mathcal{O}$-coherent and D'_W-invariant.

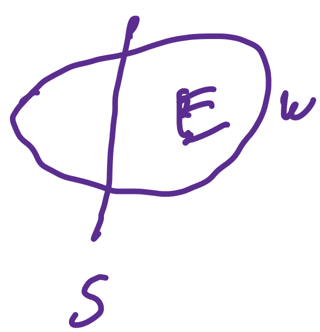
Set $E' = l_*(E'') \cap F \subset F$

$l : W \to X$ It is obviously D'_X-invariant

I claim that it ...
$\mathcal{O}$-coherent.

This follows from

Lemma. Let X be alg. var. normal, $V \subset X$ open s.t. codim $X \setminus V$ in X ≥ 1. Let $\mathcal{E}$ be a coherent $\mathcal{O}_V$-module without torsion. Then $i_*(\mathcal{E})$ is coherent $\mathcal{O}_X$-module.



$F = \hat{j}_* \mathcal{E}$

$E' \subset F|_W$

$i_* E' \cap F$

$S - (S \cap W)$

$\dim S \setminus (S \cap W) < \dim S \leq \dim X - 2$

Not coherent subsheaf N

for $E' = i_* E' \cap F =$

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$$= l_* E' \cap i_* E$$

$N \subset S$, $N \subset X \setminus W$

$E' \subset$ direct image of

$E'|_{U \cup W}$

X smooth curve $s \in X$

$U = X \setminus s$. E - smooth on U.

$F = i_* E \supset E'$ - lattice

$E_k = E / t^k E$. f. dim. space

consider action of d

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on this space and its eigenvalues with multiplicities.

$R_k = R_1 \cup (R_1 + 1) \cup (R_1 + 2), \ldots R_1 + (k-1)$.

E_k has filtration

$E = E_0 \supset E_1 \supset \ldots \supset E_k$

$R(E/E_k) = R(E_0/E_1) \cup R(E_1/E_2) \cup \ldots R(E_{k-1}/E_k)$

$E_i / E_{i+1} = t(E_{i-1}/E_i)$

$R(E_i/E_{i+1}) = R(E_{i-1}/E_i) + 1$

$[d, t] = t$

$x_1, \dots, x_n \in \mathbb{R}$

$x_1 - x_n$, $x_1+1, \dots, x_n+1$

E' defines a subset in $\mathbb{C}$

$R(E) = x_1+\mathbb{Z}, \dots, x_n+\mathbb{Z}$

$R(E)$ does not depend on choice of E'

given two choices

$\exists E'$ s.t. $0 \le \operatorname{Re} x_i < 1$ for all i