D-modules-Lecture-3

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D-
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3. Lecture 3. RS in high dimension

We have discussed the notion of RS in dimensions ≤ 1.

How to define the notion of RS in higher dimensions?

Recall, that in case of holonomic modules and complexes we could give a definition using restrictions to points. So we can define RS using restrictions to curves. Later we will discuss other approaches.

Let X be an algebraic variety. A **test curve on** X is a morphism $\nu : C \to X$, where C is a smooth curve.

Definition. . A $\mathcal{D}$-complex F on X is called RS if it is holonomic and for any test curve (C, ν) the restriction $\nu^!(F)$ is RS on the curve C.

We denote by $D_{RS}(\mathcal{D}_X)$ the category of RS-complexes (a full subcategory of $D(\mathcal{D}_X)$).

A $\mathcal{D}_X$-module is called RS if it is RS as a $\mathcal{D}_X$-complex. These modules form a full subcategory $RS(\mathcal{D}_X)$ of $M(\mathcal{D}_X)$.

Discussion – Pro and contra of this definition.

Our goal is to show the following

Theorem 3.1.

1. *Subcategory D_{RS} is a triangulated subcategory closed under extensions.*

2. *Categories of RS complexes are preserved by all functors*

$\pi^!, \pi_*, \pi_!, \pi^*, \boxplus, \mathbb{D},.$

3. *A $\mathcal{D}_X$-complex F is RS iff all its cohomology modules are RS.*

4. *The subcategory $RS(\mathcal{D}_X) \subset M(\mathcal{D}_X)$ is an abelian subcategory closed with respect to subquotients and extensions.*

3.2. RS **for smooth modules.** We know that holonomic complexes can be generated by images of smooth $\mathcal{D}$-modules. So it is natural to study the notion RS first for smooth $\mathcal{D}$-modules.

Let X be a smooth variety, E a smooth $\mathcal{D}$-module on X. We can think about E as a vector bundle with a flat connection ∇.

For any test curve $\nu : C \to X$ we see that $\mathcal{D}_C$-complex $\nu^!(E)$ up to cohomological shift coincides with the vector bundle $\nu^*(E)$ with induced connection. Hence E is RS iff it satisfies the following condition

(RS) For any test curve (ν, C) the bundle $\nu^*(E)$ on C

(RS) For any test curve (ν, C) the bundle $\nu^*(E)$ on C is RS.

Let us consider slightly more general situation.

3.2.1. *Regular singularity along a closed subset S.* Let X be a smooth algebraic variety of dimension n, $S \subset X$ a closed subset (usually it will be a divisor).

Set $U = X \setminus S$ and denote by $j : U \to X$ the open imbedding.

Let E be a smooth $\mathcal{D}_U$-module. We would like to define a notion that E is RS along the subset S.

In this situation we consider **pointed test curves** Namely, this is a pointed smooth curve (C, s) equipped with a morphism $\nu : C \to X$ such that $\nu(s) \in S$ and $\nu(C \setminus s) \subset U$.

We say that E **is** RS **along** S if it satisfies the following condition

(RS) For any pointed test curve (ν, C, s) the bundle $\nu^*(E)$ on $C \setminus s$ is RS at the point s.

In the study of smooth RS-modules important role is played by the following informal

Principle. If the condition RS holds for many pointed test curves then it holds for all pointed test curves.

3.2.2. *RS along smooth divisor S.* Let us consider the important case when X is smooth and $S \subset X$ is a smooth divisor. We denote by $\mathcal{D}_{X,S}$ the sheaf of subalgebras in $\mathcal{D}_X$ generated by $\mathcal{O}_X$ and by vector fields tangent to S.

Locally we can choose coordinate system $x_1, ..., x_n$ on X such that S is defined by equation $t = 0$, where $t = x_n$. Then the algebra $\mathcal{D}_{X,S}$ is generated by $\mathcal{O}_X$ and vector fields ∂_i for $i = 1, ..., n-1$ and $d = t\partial_n$

Let E be a smooth $\mathcal{D}_U$-module, where $U = X \setminus S$. We set $F := j_*(E)$.

Definition. 1. We call an S**-lattice in** F a coherent $\mathcal{O}_X$-submodule E' such that the restriction of E' to U coincides with E.

2. We say that the S-lattice E' is **admissible** if is $\mathcal{D}_{X,S}$ -invariant.

3. We say that the smooth $\mathcal{D}$-module E is **algebraically** RS **along** S if the sheaf F has an admissible S- lattice.

It is easy to prove the following

Lemma 3.2.3. *(i) Any two S-lattices E', E'' are (locally) t-equivalent, i.e. there exists a number N such that $E'' \subset t^{-N}E'$ and $E' \subset t^{-N}E''$*

(ii) If F has an admissible S-lattice, then any $\mathcal{O}_X$-coherent subsheaf $E' \subset F$ is contained in an admissible S-lattice.

$X = \mathbb{P}^1$, $S = \{0\}$. $U = \mathbb{A}^1 \setminus 0$

E corresponds to function

$f = e^{1/t}$

$\partial_t f = -\frac{1}{t^2} f$ — not regular sing

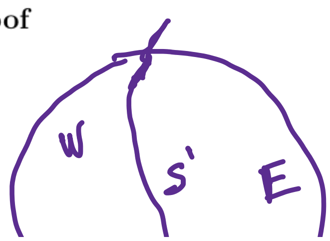
We will prove the following key criterion of RS

Proposition 3.2.4. *E is algebraically RS along S iff it is RS along S, i.e. its restriction to any test curve is RS.* pointed

Corollary 3.2.5. *Let S be a smooth divisor.*

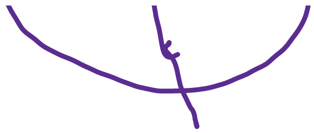
Suppose there exists an open dense subset $S' \subset S$ and its open neighborhood W in X such that the restriction of the smooth $\mathcal{D}_X$ module E to W is RS along S'. Then E is RS along S.

Proof



$F|_W \supset E'$ adm lattice

$i: W \to X$



$H = R_*(E') \cap F$

$H \subset F$ $\mathcal{O}_X$-subm

$j : U \to X$

$F = j_*(E)$

claim. (1) H is $\mathcal{D}_{X,S}$-invar

(2) is $\mathcal{O}_X$-coherent

i.e. H is an admissible on F $\Rightarrow$ E is alg RS

proof of 2.

(i) H does not have torsion.

(ii) H is coherent on U and on W

$X \setminus (U \cup W)$ has codim ≥ 1

Lemma. Let H be a qch. $\mathcal{O}_X$-module X-ired. Suppose H is without torsion and H is coherent outside of a closed subset T of codim ≥ 1. Then H is coherent.

$V = X \setminus T$ $\quad v : V \to X$

$H \quad H \subset v_*(H|V)$

Sublemma. $V \subset X$-open. R is $\mathcal{O}$-coherent $\mathcal{O}$-module without torsion

$V = X \setminus T$ codim $T \geq 1$.

Then $\underline{v_*(R)}$ is coherent

i) case $R = \mathcal{O}_V$

ii) $R \subset \mathcal{O}_V^n$

Functionals $f : R \to \mathcal{O}_V$ separate elem of R

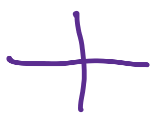
Finite number of f separate.

$R \subset \mathcal{O}_V^n$.

3.2.6. *Divisor with normal crossings.*

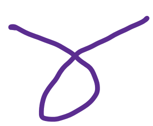
$S \subset X$

S - divisor with normal crossings if locally $\exists$ coord. system $x_1, \dots, x_n$ s.t. S is given by equation $x_n \cdot x_{n-1} \cdots x_{k} = 0$



claim. corollary holds in this case.

Example $X = \mathbb{C}^2$

S 

proof.

$\mathcal{D}_{X,S}$ - sheaf of subalgs generated by O_X and vector fields tangent to S.

Locally it is generated by $O_X, \partial_1, \dots, \partial_{n-k}$,

$d_i = x_i \partial_{x_i}$ $i = n-k+1, \dots, n$

<u>Proposition.</u> E is algebraically

... S iff it is

RS along S ...

RS along S.

Proof.

algebraic RS $\Rightarrow$ RS.

we have admissible lattice $E' < F$

consider curve $(C, 1)$ with parameter $t \to 1$

$x_i = x_i(t)$, $x_i(0) = 0$ for $i =$ $i > n, n-1, \dots n-k$.

$x_i(t) = c\, t^{x_i} \dots$ $i \dots$

claim. Vector field $t\partial_t$ on C can be extended to a vector field $\xi \subset D_{X,S}$ — ~~tangent~~ to S

This implies that if $E'|_C$ is an admissible lattice in $F|_C$

Deligne's criterion

$X \supset S$ $\quad$ E — smooth on $U = X \setminus S$