D-modules-Lecture-4

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D-
modules-

4. Lecture 4. Main theorem about regular singular $\mathcal{D}$-complexes

Let us denote by $I(X)$ the set of isomorphism classes of simple holonomic $\mathcal{D}_X$-modules. Any holonomic $\mathcal{D}_X$-module F has finite length, so it defines a finite subset $Cont(F) \subset I(X)$ (content of the module F). If F is a holonomic $\mathcal{D}_X$-complex we define its content $Cont(F) := \cup_i Cont(H^i(F)$ (union of contents of its cohomologies.

Our goal is to show the following

Theorem 4.1.

1. *Subcategory D_{RS} is a triangulated subcategory closed under extensions.*

2. *Categories of RS complexes are preserved by all functors*

$\pi^!, \pi_*, \pi_!, \pi^*, \boxplus, \mathbb{D}$.

3. *A holonomic $\mathcal{D}_X$-complex is RS iff $cont(F)$ consists of RS-modules. In particular. this holds iff its cohomology complexes are RS.*

4. *The subcategory $RS(\mathcal{D}_X) \subset M(\mathcal{D}_X)$ is an abelian subcategory closed with respect to subquotients and extensions.*

Discussion

4.2. **First properties. Step 1.** Categories RS are closed with respect to extensions.

Step 2. Functors $\pi^!$ preserve RS. Functor $\boxtimes$ preserves RS.

Step 3. If $j : Z \to X$ is a locally closed imbedding then the functor j_* preserves RS.

If $\pi : X \to Y$ is a finite morphism then the functor π_* preserves RS.

Step 4. Let Z be a smooth irreducible affine variety, E a smooth RS $\mathcal{D}$-module on Z and $j : Z \to X$ be a locally closed embedding. Then the $\mathcal{D}_X$-module $j_*(E) \subset M(\mathcal{D}_X)$ is RS. We call such a module an elementary RS $\mathcal{D}_X$-module.

Every RS-complex is glued from elementary RS-modules (and their homological translates).

X, F $\qquad$ Y, H

$X \times Y$ $\quad$ $F \boxtimes H$

$v : C \to X \times Y$

$v^!(F \boxtimes H) = (v_1)^! F \overset{D}{\otimes} (v_2)^! H$

$S = \{Y_i$ - loc. closed subsets of $X\}$.

$j_i : X_i \hookrightarrow X$

$F_i = (j_i)_* j_i^! F$

F is glued from F_i

F – holonomic.

Then we can choose X_i s.t. X_i are smooth affine divisors

$E_i = j_i^!(F)$ is smooth.

F - RS, $E_i = j_i^! F$ is smooth RS.

Elementary RS-module

is a module $j_!(E)$,

$j: U \to X$ is affine, smooth, curves

E is RS module on U.

Every RS complex on X is glued from elem modules and their cohom. shifts.

$j: U \to X$, U-affine $\Rightarrow$ j - affine

loc. closed embed.

$\Rightarrow j_! E$ is a D-module.

4.3. **Good compctifications.** Our proof uses deep results from Algebraic Geometry related to resolution of singularities of algebraic varieties over a field of characteristic 0.

Definition. Let U be an irreducible smooth affine algebraic variety. A **good compactification** of U is a smooth projective variety $\bar{U}$ equipped with an open imbedding $j : U \to \bar{U}$ such that the complement $S = \bar{U} \setminus U$ is a divisor with strict normal crossings.

Theorem 4.4. *Any smooth affine irreducible* smooth *variety*

U has a good compactification..

Moreover, if we have a morphism $\pi : U \to P$, where P is a projective variety, then we can find a good compactification $\bar{U}$ such that the morphism π extends to a morphism $\pi' : \bar{U} \to P$.

Good compactifications are useful since for them we have very explicit description of some RS modules.

Let $X = \bar{U}$ be a good compactification of U and $S \subset X$ be a divisor with normal crossings.

Let F be a $\mathcal{D}_X$-module. We say that F is algebraically RS with respect to S if it is a union of $\mathcal{O}_X$-coherent $\mathcal{D}_{X,S}$-modules.

Claim. ***1.*** *Let E be a smooth RS $\mathcal{D}_U$-module, $F = j_*(E)$. Then F is a $\mathcal{D}_X$-module, such that it and all subquotients of it are algebraically RS. In particular, they are all RS.*

2. *Let ν be the natural morphism $\nu : j_!(E) \to j_*(E)$. Then the modules $L(S, E) = Im(\nu)$ and $Coker(\nu)$ are RS.*

$\dots \mathcal{D}_X$ – subsheaf of
algebras gener. by $\mathcal{O}_X$ and
vector fields tangent to S.

$j : U \to X \qquad F = j_* E$

4.5. **direct image preserves** RS. We would like to show that the functor of direct image preserves RS.

Step 5. Reduction to the following

Claim. *X is a good compactification of a surface U, $F = j_* E$ where E is a smooth RS $\mathcal{D}_U$-module, $p : X \to Y$ a morphism of X onto a smooth projective curve Y. Then the complex $p_*(F)$ on Y is RS.*

Proof of reduction.

$\pi : X \to Y$, F – RS $\mathcal{D}_X$-comp.

Want $H = \pi_* F$ is RS

(i) Can assume X, Y are affine

(ii) Can assume $X = \mathbb{A}^n$, $Y = \mathbb{A}^1$

$\pi : X \to Y$ projection

(iii) $X \to S \quad X = \mathbb{A}^{n+1} \dots \mathbb{A}^n$

(iii) X ... $X - A \longrightarrow Y$ ...

$X = A \times Y$ π - projection

(iv) $R\pi_* F$ is RS,

$\nu : C \longrightarrow Y$ $\nu^! \pi_* F$ is RS.

$\nu^! \pi_* = \pi_* (\nu')^!$

$C \times A \to X$

$C \longrightarrow Y$

$X = A \times Y$, ~~...~~ C is a curve

$Y \supset C$

π - projection

$U \subset X$ open, E on U - RS smooth

$\&$ Y - smooth curve.

$\pi : U \longrightarrow Y$ $\pi_* E$ is RS.