LECTURE 5 IN D-MODULES II - PROPERTIES OF ALGEBRAIC-RS MODULES

Let U be a smooth quasi-projective algebraic variety, and X be its good compactification. This means that X is projective and $S := X \setminus U$ is a divisor with strict normal crossings.

Definition 1. Let $\mathcal{D}_{X,S} \subset \mathcal{D}_X$ denote the sheaf of subalgebras generated by $\mathcal{O}_X$ and by vector fields tangent to S.

A coherent $\mathcal{D}_X$-module $\mathcal{F}$ is called *algebraic RS* (with respect to U) if its restriction $\mathcal{F}|_U$ is smooth, and $\mathcal{F}$ is a union of $\mathcal{O}_X$-coherent $\mathcal{D}_{X,S}$-submodules.

In this lecture we will prove several claims on algebraic RS-modules that were left from before.

Exercise 2. The category of algebraic RS-modules is closed w.r. to subquotients.

Proposition 3. *Let $\mathcal{E}$ be a smooth RS $\mathcal{D}_U$-module. Then $j_*\mathcal{E}$ is algebraic RS (with respect to U).*

In the proof we will use the following statements.

Exercise 4. $\mathcal{F}$ is algebraic RS if and only if every coherent $\mathcal{O}_X$-submodule of $\mathcal{F}$ generates an $\mathcal{O}_X$-coherent $\mathcal{D}_{X,S}$-submodule.

Lemma 5. *The proposition holds if S is smooth, and without the assumption that X is projective.*

This lemma will be proven in later weeks.

Exercise 6. Let Y be an algebraic variety, and $V \subset Y$ and open subset such that $\dim Y \setminus V \leq \dim Y - 2$. Let $\mathcal{H}$ be an $\mathcal{O}_Y$-module without torsion.

(i) If $\mathcal{H}|_V$ is coherent then $\mathcal{H}$ is coherent.
(ii) Let $\mathcal{H}_1 \subset \dots \mathcal{H}_i \subset \mathcal{H}_{i+1} \subset \cdots \subset \mathcal{H}$ be an increasing sequence of coherent submodules. If the sequence of restrictions $\mathcal{H}_i|_V$ stabilizes then so does the original sequence $\mathcal{H}_i$.

Proof of Proposition 3. Let Z denote the singular locus of S, and let $V := X \setminus Z$. Let $\mathcal{F}' \subset \mathcal{F} := j_*\mathcal{E}$ be an $\mathcal{O}_X$-coherent submodule, and let $\mathcal{H} \subset \mathcal{F}$ be the $\mathcal{D}_{X,S}$-submodule generated by $\mathcal{F}'$. Then $\mathcal{H}|_V$ is $\mathcal{O}_V$-coherent by the smooth case - Lemma 5. Thus by Exercise 6 $\mathcal{H}$ is $\mathcal{O}_X$-coherent. ☐

Corollary 7. *Let $\nu : j_!\mathcal{E} \to j_*\mathcal{E}$ be the natural morphism. Then the modules $L(S,\mathcal{E}) :=$* $\mathrm{Im}(\nu)$*, and* $\operatorname{Coker}\nu$ *are algebraic RS.*

Proposition 8. *Let Z be a smooth affine curve, $Y = Z \times \mathbb{A}^1$, $V \subset Y$ open dense, $p : V \to Z$ the natural projection. Let $\mathcal{E}$ be a smooth RS $\mathcal{D}_V$-module. Then $p_*\mathcal{E}$ is RS on Z.*

To prove this we may restrict to an open subset Z' of Z, and its preimage U in V. We can find a good compactification X of U, mapping onto the completion C of Z', such that the inverse image of $T = C \setminus Z'$ is contained in $S = X \setminus U$, and that T contains the image of the singular locus of S. By adding some points of C to T, and their preimages to S, if needed, we may further arrange that any component of S which maps onto C is an unramified covering outside its intersection with the singular set of S. Now let $j : U \hookrightarrow X$ denote the inclusion and $\pi : X \to C$ denote the projection. Let $\mathcal{F} := j_*\mathcal{E}$. We know that $\mathcal{F}$ is algebraic RS. Thus it is enough to prove the following proposition.

Proposition 9. *Let $\mathcal{F}$ be a $\mathcal{D}_X$-module that is algebraic RS (with respect to U). Then $\pi_*\mathcal{F}$ is algebraic RS on C (with respect to Z).*

To prove this proposition we need to construct lattices on the cohomologies of $\pi_*\mathcal{F}$. We do so by constructing pushforward for $\mathcal{D}_{X,S}$-modules. We will show that this construction will preserve $\mathcal{O}$-coherence since π is projective.

Let $\theta_{X,S}$ denote the vector fields on X that at points of S are tangent to S. Similarly, let $\theta_{C,T}$ denote vector fields on C that vanish at the points of T. At each point $x \in X$, the differential $d\pi$ maps $T(X)_x$ into $T(C)_c$. Thus we have a canonical morphism of $\mathcal{O}_X$-modules $\nu : \theta_{X,S} \to \pi^*\theta_{C,T}$.

Lemma 10. *ν is onto and its kernel is the module $\theta_{X/C}$ consisting of germs of vector fields tangent to the fibers of π.*

Proof. The statement on the kernel is easy. Let us prove that ν is onto. This statement is local on C. Over Z it is clear, since on the preimage of Z π is a submersion. Now let $x \in \pi^{-1}(T)$. We may choose a local coordinate t on C around $\pi(x)$, and local coordinates u, v on X around x such that $t \circ \pi = a(u,v)u^m v^n$, where $a(u,v)$ is regular and invertible around x, and $m > 0, n \geq 0$. Around x, the $\mathcal{O}_{\mathbb{C}}$-module $\theta_{C,T}$ is spanned by $t\partial_t$. Thus it suffices to lift $t\partial_t$, i.e. to find $\xi \in \theta_{X,S}$ such that $\xi(t \circ \pi) = t \circ \pi$. Note that around x, the $\mathcal{O}_X$-module $\theta_{X,S}$ contains $u\partial_u$. Take

$$\xi = (m + a^{-1}(u\partial_u a))^{-1} u\partial_u$$

Since

$$u\partial_u(a(u,v)u^m v^n) = (m + a^{-1}(u\partial_u a))a(u,v)u^m v^n),$$

we have $\xi(t \circ \pi) = \xi(a(u,v)u^m v^n) = a(u,v)u^m v^n = t \circ \pi$, so ξ is a lift of $t\partial_t$. ☐

Proof of Proposition 9. For every $\xi \in \theta_{X,S}$, there exist $u_i \in \mathcal{O}_X$ and $\xi_i \in \theta_{C,T}$ by

$$\nu(\xi) = \sum_i u_i \otimes \pi^{-1}\xi_i$$

For every left $\mathcal{D}_{C,T}$-module M, let $\pi^0(M)$ denote its pullback to X as an $\mathcal{O}$-module, with a $\mathcal{D}_{X,S}$-module structure given by

$$\xi(f \otimes m) = \xi f \otimes m + \sum_i f \cdot u_i \otimes \pi^{-1}(\xi_i m)$$

Let $\mathcal{D}_{X,S\to C,T} := \pi^0(\mathcal{D}_{C,T})$, endowed with the natural left $\mathcal{D}_{X,S}$-module structure. For every right $\mathcal{D}_{X,S}$-module N define $\pi_* M := R\pi_+(N \otimes^L_{\mathcal{D}_{X,S}} \mathcal{D}_{X,S\to C,T})$, where π_+ refers to the morphism of ringed spaces $(X, \pi^{-1}\mathcal{D}_{C,T}) \to (C, \mathcal{D}_{C,T})$, and R denotes the right derived functor. It is enough to show that π_* maps $\mathcal{O}_X$-coherent $\mathcal{D}_{X,S}$-modules to complexes with $\mathcal{O}_C$-coherent cohomologies. To do this note that the module $\mathcal{D}_{X,S\to C,T}$ admits a locally free resolution

$$0 \to \theta_{X/C} \otimes_{\mathbb{C}} \mathcal{D}_{X,S} \to \mathcal{D}_{X,S} \to \mathcal{D}_{X,S\to C,T} \to 0$$

Thus

$$N \otimes^L_{\mathcal{D}_{X,S}} \mathcal{D}_{X,S\to C,T} = \{N \otimes_{\mathbb{C}} \theta_{X/C} \to N\}, \text{ with differential } u \otimes \xi \mapsto u\xi$$

Since it consists of $\mathcal{O}_X$-coherent modules and π is projective, the direct image of this complex has $\mathcal{O}_C$-coherent cohomologies. □

Proposition 11. *Let U be a smooth (quasi-projective) algebraic variety, and X be a good completion of U. Let $\mathcal{F}$ be $\mathcal{D}_X$-module. If $\mathcal{F}$ is algebraic RS with respect to U then $\mathcal{F}$ is RS.*

Proof. We have to show that for every smooth projective curve C, and every morphism $\nu : C \to X$, the inverse image $\nu^*\mathcal{F}$ is RS. For this we may assume that $V := \nu^{-1}(U)$ is open dense in C, and we have to construct a lattice around every point $c \in C \setminus V$. We construct the lattice by pulling back an $\mathcal{D}_X$-coherent $\mathcal{D}_{X,S}$ submodule $\mathcal{H} \subset \mathcal{F}$ that satisfies $\mathcal{H}|_U = \mathcal{F}$. To show that it is a lattice, let t be a local coordinate at c, and $d = t\partial_t$. We have to show that $d \in \nu^*\mathcal{D}_{X,S}$.

If S is smooth at $\nu(c)$, let x be a local coordinate at $\nu(c)$ transversal to S. Then $x = a(t)t^n$, where $n > 0$ and $a(t)$ is invertible near 0, and $d = \xi := (n + a^{-1}(t)(t\partial_t a(t))x\partial_x$. indeed, $\xi x = (t\partial_t a(t) + na(t))t^n$, and thus $\xi = d = t\partial_t$.

If S is not smooth at $\nu(c)$, it still has normal crossings. Suppose it has 2 components near $\nu(c)$, with local coordinates x and y. Then $x = a(t)t^n$, and $y = b(t)t^m$. Take

$$\xi := (n + a^{-1}(t)(t\partial_t a(t))x\partial_x + (m + b^{-1}(t)(t\partial_t b(t))y\partial_y$$

□