D-modules-Lecture-6

Monday, 3 May 2021 10:09

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5. Lecture 6. Regular connection along smooth divisor

Let X be a smooth irreducible variety of dimension n, $S \subset X$ a smooth irreducible closed divisor, $U = X \setminus S$, $j : U \to X$ open imbedding.

Let E be a smooth $\mathcal{D}$-module on U and $F = j_*(E)$ the corresponding $\mathcal{D}$-module on X.

Suppose we know that for many curves C transversal to S the following condition holds

$(RS)$the restriction of E to C is RS at the intersection $C \cap S$.

We would like to prove that then E is algebraically RS along S.

This would imply that the condition (*) holds for all curves C, I will use all the time the following lemma from Algebraic Geometry

Lemma 5.0.1. *Let X be an irreducible algebraic variety of dimension n, F be a quasicoherent $\mathcal{O}_X$-module without torsion. Let and $M \subset F$ its quasicoherent submodule.*

(i) Suppose that M is coherent on the open subset $X \setminus T$, where T is a closed subset of codimension > 1. Then M is coherent everywhere.

(ii) Let $M_1 \subset M_2 \subset ....$ be a sequence of coherent submodules of F. Suppose it stabilizes on the subset $X \setminus T$. Then it stabilizes on X.

(iii) Suppose $M \subset F$ is a coherent $\mathcal{O}_X$ submodule. Consider the collection of all intermediate coherent submodules $M \subset N \subset F$ such that the support of N/M has codimension > 1.

Then the sum M' of all these modules is coherent and the support of M'/M has codimension > 1.

We call this module M' the **enhancement** of M.

We say that the submodule $M \subset F$ is **enhanced** if $M' = M$

Example 5.0.2. $X = A^2$, $F = K(X)$ - sheaf of rational functions on X, $M \subset \mathcal{O}_X$ the subsheaf of functions that vanish at 0. In this case $M' = \mathcal{O}_X$ is an enhanced submodule.

Let $\mathcal{D}_{X,S}$ be the subsheaf subalgebras of $\mathcal{D}_X$ generated by $\mathcal{O}_X$ and vector fields tangent to S. The condition ated by $\mathcal{O}_X$ and vector fields tangent to S. The condition **Algebraic** RS is as follows

(Alg.RS) For any lattice $L \subset F$ the sheaf $\mathcal{D}_{X,S} \cdot L$ is $\mathcal{O}_X$-coherent, i.e. it is an admissible lattice.

In case of a curve this is one of the definitions of RS.

The lemma above immediately implies that we can pass to an open subset, i.e. in the proof we can replace X by an open subset intersecting S.

Using this we will assume that X has a coordinate system $x_1, ..., x_n$ such that S is defined by the equation $t = 0$, where $t = x_n$.

In this case the algebra $\mathcal{D}_{X,S}$ is generated by $\mathcal{O}_X$ and vector fields ∂_i for $i = 1, ..., n-1$ and $d = t\partial_n$.

Our goal is the following

Theorem 5.1. *Consider the morphism* $p : X \to A^{n-1}$ *defined by coordinates* $(x_1, ..., x_{n-1})$ *and consider the family of curves* C_a *on* X *defined by fibers of this morphism.*

Suppose we know that for a Zariski dense subset of points $a \in A^{n-1}$ *the restriction of the smooth* $\mathcal{D}$*-module*

E to the curve C_a is RS at points of S.

Then the $\mathcal{D}$-module E is algebraically RS along S.

Proof.

Step 1.

Let M be a lattice in F. Consider an increasing sequence of lattices $M = M_1 \subset M_2 \subset ...$ in F, where $M_{i+1} = M_i + dM_i = M_i + \mathcal{O}_X dM_i$. Let us show that this sequence stabilizes.

Set $N_i = M_i/M_{i-1}$. Then it is easy to check that $d : N_i \to N_{i+1}$ is a morphism of $\mathcal{O}_X$-modules. it is also clearly an epimorphism.

Since the module N_1 is coherent it is Noetherian. This implies that $d : N_i \to N_{i+1}$ is an isomorphism for large i.

Choose the index i with this property. Then modules $N_i, N_{i+1}...$ are isomorphic to some module N.

The closed subset $T = supp(N)$ is contained in S. If it is strictly less than S we replace S by an open subset $S \setminus T$ and see that on this subset modules M_i stabilize. Then they stabilize everywhere.

If $T = S$ then we can find an open subset $V \subset A^{n-1}$ such that N is a free non zero module over the algebra $\mathcal{O}_V$.

This implies that for any point $a \in V$ when we restrict the exact sequence $0 \to M_k \to M_{k+1} \to N \to 0$ to the curve C_a we get again an exact sequence $0 \to M_{a,k} \to M_{a,k+1} \to N_a \to 0$.

This means that the sequence of modules $M_{a,k} \subset F_a$ on the curve C_a – does not stabilize.

Hence the restriction of the smooth $\mathcal{D}$ -module E to the curve C_a is not RS – a contradiction.

Warning.

If $\exists$ one curve C transverse to S s.t. $E|_C$ is RS –

- is not enough for E-being RS

Counter example: E-smooth D-module corresponding to function $f = \exp(\frac{x}{y})$, S is given by $y=0$

Step 2. Using step 1 we can assume that our lattice $M \subset F$ is d-invariant. We also assume that it is enhanced.

Let K be the field of rational functions on the divisor S. To any lattice $M \subset F$ we assign a polynomial $P_M \in K[t]$.
Namely consider the $\mathcal{O}_S$-module M/tM with the operator d, pass to the corresponding K-vector space V_M and denote by P_M the characteristic polynomial of the operator $d : V \to V$.

It is easy to see that $P_{tM}(t) = P_M(t + 1)$- this follows from the relation $[d, t] = t$.

From this we get the following

Lemma 5.1.1. *There exists a number* N *such that for any* $k > N$ *polynomials* P_M *and* $P_{t^{-k}M}$ *are relatively prime.*

In particular, there are no non zero morphisms $\nu : V_M \to V_{t^{-k}}M$ *that commute with the operator* d*.*

Corollary 5.1.2. *Let* $M \subset F$ *be an enhanced* d*-invariant lattice. Then* $\mathcal{D}_{X,S}(M)$ *is contained in* $t^{-N}M$*.*

Of course this corollary implies the theorem.

Proof of the corollary.

We assume that X is affine. Let $D' \subset \mathcal{D}_X$ be the subalgebra generated by $\mathcal{O}_S$ and operators ∂_i for $i < n$. All these operators commute with the operator d

For any $\delta \in D'$ there exists a number k (that depends on δ), such that the lattice $L = t^{-k}M$ contains δM.

We choose k minimal with this property. Our goal is to show that $k \leq N$ – this will prove the corollary.

We fix k. we can assume that δ is an element of minimal degree such that $\delta M \subset L$.

This implies that the operator $\nu : V(M) \to V(L)$ given by $\nu(m) = \delta(m) \bmod tL$ is K-linear.

If $k > N$ such operator should be 0.

If $\nu = 0$ then, since the lattice L is enhanced, we get that $\delta M \subset tL$. This contradicts to the minimal choice of k.

$\delta : M \to L = t^{-k} M$

Let $f \in \mathcal{O}(S)$.

$\delta' = \delta f - f\delta$ is operator of order $<$ ord δ.

Hence $\delta' M \subset tL$

$\delta : M \to V_L$ is $\mathcal{O}_t$-linear.

By assumption $\delta : M/tM \to V_L$ is not 0.

Since M is enhanced $\delta : V_M \to V_L$ is not 0.

Hence $k \leq N$ since δ commutes with d.