D-modules-Lecture-7

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D-
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7. LECTURE 7. RIEMANN - HILBERT CORRESPONDENCE.

In this lecture we will consider algebraic geometry over the field $\mathbb{C}$ of complex numbers.

Given a smooth algebraic variety X over $\mathbb{C}$ we can consider the set of points $X(\mathbb{C})$ as a topological space in usual – analytic – topology. We denote this space by X_{top}.

This space in fact is an oriented manifold. Our general goal is to understand how to connect topological invariants of this space with algebraic invariants of the variety X. Let me briefly discuss what we can expect.

Let $Sh(X_{top}$ denote the category of sheaves of complex vector spaces on X_{top}. We will see that many invariants defined in terms of this category have an algebraic counterpart.

To study these invariants we will use an intermediate object – the complex analytic variety X_{an}. This is the topological space X_{top} equipped with the sheaf O_{an} of analytic functions.

7.1. **Local systems.**

Definition. A local system (of complex vector spaces) on the topological space X_{top} is a sheaf of complex vector spaces that is locally isomorphic to the constant sheaf $\mathbb{C}^n$ for some n.

We denote by $LocSys(X_{top})$ the full subcategory of local systems of the category $Sh(X_{top})$.

Remark. Local systems and representations of the fundamental group.

X - connected manifold, $x_0 \in X$
$\pi_1(X, x_0)$
L - local system on X
$L(x_0)$ - fiber at x_0 then
we get an action of $\pi_1(X, x_0)$ on $L(x_0)$.

$$Loc(X) \simeq Rep(\pi_1(X, x_0))$$

The category $LocSys(X_{top})$ can be described in algebraic terms. First state the following easy result.

Proposition 7.1.1. *Let $X_{an} = (X_{top}, O_{an})$ be a smooth complex analytic variety over $\mathbb{C}$. The category $LocSys(X_{top})$*

of local systems on the space X_{top} is canonically equivalent to the category $Smooth(X_{an})$ of analytic vector bundles E on X_{an} equipped with a holomorphic flat connection ∇.

The mutually inverse functors here are given by

$L \mapsto O_{an} \otimes_{\mathbb{C}} L$

$(E, \nabla) \mapsto$ sheaf of flat sections of E.

Let X be a projective smooth algebraic variety over $\mathbb{C}$. We will see later that in this case the category of algebraic vector bundles on X is canonically equivalent to the category of analytic vector bundles on the analytic variety X_{an}. From this it is easy to deduce the following

Corollary 7.1.2. *Let X be a smooth projective algebraic variety over $\mathbb{C}$. Then the category $LocSys(X_{top})$ is canonically equivalent to the category $Smooth(\mathcal{D}_X)$ of smooth $\mathcal{D}_X$-modules.*

In fact, this is true for any complete smooth variety. We will prove this later and also we will show how to compute the cohomology of local systems in algebraic terms.

However, in the case when X is not complete the situation is more complicated.

Example 7.1.3. Let $X = A^1$. consider two smooth $\mathcal{D}_X$-modules of dimension 1 over $\mathcal{O}_X$

$M = \mathcal{D}_X/\mathcal{D}_X \cdot \partial_t$ and $N = \mathcal{D}_X/\mathcal{D}_X \cdot (\partial_t - 1)$.

These modules are generated by functions $f = 1$ and $h = \exp(t)$.

Analytically these modules are isomorphic, and they both correspond to the trivial local system on X_{top}. But as algebraic $\mathcal{D}_X$-modules they are not isomorphic.

7.2. **How to state the Riemann–Hilbert correspondence.** The classical Riemann-Hilbert correspondence is the following

Theorem 7.3. *Let X be a smooth algebraic curve. Then the category $LocSys(X_{top})$ is canonically equivalent to the category of smooth RS $\mathcal{D}_X$-modules.*

This theorem has natural generalization for an arbitrary smooth algebraic variety X – in fact, it is formulated in exactly the same way.

However, in 1976 Kashiwara conjectured that some much more general claim has to be true. Kashiwara and Kawai have proven some form of this conjecture, but their proof was very heavy.

The reason was that they worked in the framework of $\mathcal{D}$-modules on complex analytic varieties, and this theory is much less structured than the theory of algebraic $\mathcal{D}$-modules. For example, for such modules there is no natural general notion of direct image.

$j : \mathbb{C}^* \to \mathbb{C}$

$j_* : \mathcal{M}(\mathcal{D}_{\mathbb{C}^*}) \to \mathcal{M}(\mathcal{D}_{\mathbb{C}})$

We will work in algebraic situation. Let us first introduce the objects that appear in RH correspondence.

7.4. **Constructible sheaves.** Let X be an algebraic variety. Let us remind that a **stratification** of X is a finite collection $\mathcal{S} = (S_i)$ of smooth algebraic subvarieties $S_i \subset X$ - they are called strata of $\mathcal{S}$ – such that they are disjoint and cover X.

Usually one ads a condition that the closure of every stratum is a union of strata and some other conditions. However for our our purposes this is not important, since any stratification can be refined to a stratification satisfying these conditions.

Let X_{top} be the topological space corresponding to XR and $Sh(X)$ the category of sheaves of complex vector spaces on X_{top}.

This category is too large. We will introduce some natural subcategory of it – the category of **constructible** sheaves, that better describes the geometry.

Definition. A sheaf $F \in Sh(X)$ is called **constructible** if there exists a stratification $\mathcal{S} = (S_i)$ of the algebraic variety X such that the restriction of F to any strata S_i is a local system on $(S_i)_{top}$.

We denote by $Con(X) \subset Sh(X)$ the full subcategory of constructible sheaves.

Note that this definition refers to the algebraic variety X, and not only to the topological space X_{top}.

7.5. **Derived category of constructible complexes.** As usual we have to pass from sheaves to derived category.

We denote by $D(X_{top})$ the bounded derived category of the category $Sh(X)$. A complex $F^{\cdot}$ in this category is called **constructible** if all it cohomology sheaves are constructible.

We denote by $D_{con}(X)$ the full subcategory of $\mathcal{D}(X_{top})$ of constructible complexes.

Thus to every algebraic variety X over $\mathbb{C}$ we have assigned a triangulated category $D_{con}(X)$. This is one of the main objects of study in algebraic geometry.

We will see that this assignment has six Grothendieck functors (in fact, they originally were defined in this situation). Namely, for every morphism $\pi : X \to Y$ of algebraic varieties we will describe exact functors

$\pi_*, \pi_! : D_{con}(X) \to D_{con}(Y)$

$\pi^!, \pi^* : D_{con}(Y) \to D_{con}(X)$

Contravariant functor of Verdier duality $\mathbb{D} : D_{con}(X) \to D_{con}(X)$

Exterior tensor product functor $\boxtimes : D_{con}(X) \times D_{con}(Y) \to D_{con}(X \times Y)$

We will define a functor $\Omega : D(X) \to D(X_{top})$ and prove the following statements

1. Functor Ω defines an equivalence of triangulated categories $\Omega : D_{RS}(\mathcal{D}_X) \to D_{con}(X)$

2. Functor Ω is compatible with six Grothendieck functors

3. Let E be a smooth RS $\mathcal{D}_X$-module on a smooth variety X of dimension n. Let $L(E)$ be the corresponding local system on the topological space X_{top}.

Then $\Omega(E) = L(E)[n]$ (cohomological shift).

This is the Kashiwara's version of the RH correspondence.

Explanation why we should expect a cohomological shift.

Functors in topological situation.

X nice top. space.

$Sh(X)$ - sheaves of complex vector spaces

$\pi : X \to Y$ continuous map

$\pi_\circ (Sh(X)) \to Sh(Y)$

$\pi_\circ (F)(U) = F(\pi^{-1}(U))$.

Left exact functor.

$\pi_* = R\pi_\circ : D(X) \to D(Y)$

$D(X) = D^b(Sh(X))$.

Functor $\pi_\cdot$ has left adjoint

$\pi^\cdot : Sh(Y) \to Sh(X)$

$\pi^\cdot$ is exact.

$\pi^* = R\pi^\cdot : D(Y) \to D(X)$.

$\pi_! : Sh(X) \to Sh(Y)$

$\mathcal{F}$ on X

$\pi_*(\mathcal{F}) = \{\xi \in \mathcal{F}(U)\}$.

$| \, |^{\text{exact}}$

$\to Y$

$\pi_{\cdot !}(\mathcal{F})(U) = \{\xi \in \mathcal{F}(\pi^{-1}(U))\}$

supp $\xi \to U$ is a proper map.

$\pi_{\cdot !} : Sh(X) \to Sh(Y)$

$\pi_{\cdot !}$ is left exact

$\pi_! = R\pi_{\cdot !}$

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canonical morphism

$\pi_! \to \pi_*$

1) isomorphism if π-proper

2) $j : X \to Y$ open embedding.

$j_!$ - extension by 0.

$\pi^! : D(Y) \to D(X)$ is a right adjoint functor for $\pi_!$

$(\pi \iota)^! = \iota^! \pi^!$

$i: X \hookrightarrow Y$ closed embeddings

$\Gamma_X(H) = \{ s \in H \mid \operatorname{supp} s \subset X \}$.

$\Gamma_X: Sh(Y) \to Sh(X)$.

Left exact.

$i^! = R\Gamma_X$

$p: X = Y \times \mathbb{R}^n \to Y$

$p^!(H) = H \boxtimes \mathbb{D}(\mathbb{R}^n)$

$\mathbb{D}(\mathbb{R}^n)$ is a constant sheet on $\mathbb{R}^n$ is dim $-n$ normalizers by

$p: \mathbb{R}^n \to pt$

$p_!(\mathbb{D}(\mathbb{R}^n)) \cong \mathbb{C}$

$\mathbb{D}: D(X) \to D(X)$

$\mathbb{D}ual(X) := p^!(\mathbb{C}_{pt}) \in D(X)$

$p: X \to pt$

$\mathbb{D}(F) = R\mathcal{H}om(F, \mathbb{D}ual(X))$

$\mathbb{D}ual(X) = p^!(\mathbb{D}ual(pt))$

Theorem.

If $\pi: X \to Y$ maps of alg. varieties then

$\pi_*, \pi_!, \pi^*, \pi^!$ preserve constructible complexes

D, $\boxtimes$ preserve constructible

$X \subset Y$ open $i: X \hookrightarrow Y$

F is in $D(Y)$

$i^! F$ lives in triangle

F $j_*(F|_{Y-X})$

$\Omega: D(P_X) \to D(X_{top})$.