D-modules-Lecture-8

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D-modules-

8. Lecture 8. Proof of Riemann - Hilbert correspondence

8.1. Construction of the functor Ω.

8.1.1. *DeRham complex of a $\mathcal{D}_X$-module.* Let X be a smooth algebraic variety of dimension n. Starting from a DX-module M we construct a complex of sheaves $DR(M)$ on X as follows.

We set $DR^k(M) := M \otimes_{\mathcal{O}_X} \Omega^{k+n}(M)$ – this is a sheaf of $\mathcal{O}_X$-modules. The differential $d : DR^k(M) \to DR^{k+1}(M)$ is defined by standard formulas for DeRham complex - this is a differential operator of degree 1.

Note that this is an exact functor $DR : M(\mathcal{D}_X) \to Com(Sh(X))$. In fact its target category can be described slightly more precisely.

Let us denote by $M(\mathcal{O}_X)^d$ the additive category whose objects are quasi-coherent $\mathcal{O}_X$-modules and morphisms are differential operators. Then the functor DR is a functor $DR : Com(M(\mathcal{D}_X) \to Com(M(\mathcal{O}_X)^d)$.

$\xi \in M$ $\quad d\xi = \sum \frac{\partial \xi}{\partial x_i} dx_i$

$DR(M)$ in degrees $-n$ to 0

Claim. *(i) The complex $DR(\mathcal{D}_X)$ gives a natural locally projective resolution of the module ω in the category of right $\mathcal{D}_X$-modules.*

(ii) For any $\mathcal{D}_X$-module M we have a canonical isomorphism $DR(M) = DR(\mathcal{D}_X) \otimes M$ (tensor product over $\mathcal{D}_X$)

In other words, the complex $DR(M)$ in derived category represents the derived tensor product over $\mathcal{D}_X$

$M \mapsto \omega \otimes^L M$

Using the bi-complex construction we extend the functor DR to the exact functor on the categories of complexes

$DR : Com(M(\mathcal{D}_X)) \to Com(M(\mathcal{O}_X)^d)$.

8.1.2. *Functor An of analytic extension.* . Given a smooth algebraic variety X we have a natural functor $An : M(\mathcal{O}_X) \to M(O_{an})$. It is locally given by tensor product $M \mapsto O_{an} \otimes M$ (tensor product over $\mathcal{O}_X$).

By definition this functor transforms $\mathcal{O}_X$-linear morphisms into O_{an} linear. But it is also clear that it maps differential operators into differential ones. Thus we can consider this functor as a functor $An : M(\mathcal{O}_X)^d \to Sh(X_{top})$.

Now we define an exact functor $\Omega : Com(M(\mathcal{D}_X)) \to Com(Sh(X_{top}))$ as a composition of functors DR and An.

$\Omega(M) = An(DR(M))$.

We consider this functor as a functor $\Omega : D(\mathcal{D}_X) \to D(Sh(X_{top}))$

8.2. **Properties of the functor Ω.**

Elementary properties.

1. Functor Ω commutes with restriction to an open subset.

2.. Functor Ω commutes with functor $\pi^!$ if π is a projection $X = Y \times Z \to Y$ with a smooth fiber Z.

3. If M is a smooth $\mathcal{D}_X$-module then $\Omega(M) = Loc(M)[n]$, where $n = \dim(X)$.

Key property 4.

For any morphisms $\pi : X \to Y$ there exists a functorial morphisms of functors $i : \Omega \circ \pi_* \to \pi_* \circ \Omega$. If π is projective this morphism is an isomorphism.

This follows from the basic result of GAGA. We will discuss this later.

5. Let $j : U \to X$ be an open embedding such that $D = X \setminus U$ is a divisor with normal crossings.

Let M be a smooth RS $\mathcal{D}$-module on U.

Then we have natural isomorphisms $j_*(\Omega(M)) \to \Omega(j_*(M))$

This is proven by explicit computations in coordinates.

Functor Ω commutes with all functors on the category $D_{RS}(\mathcal{D}_X)$

First we have to show that the functor DR on category $D_{RS}(\mathcal{D}_X)$ commutes with six Grothendieck functors.

Step 1. On subcategory $D_{RS}(\mathcal{D}_X)$ the functor Ω commutes with π_*.

It is enough to prove this for elementary modules. Hence it is enough to prove this for a smooth RS-module E. Using resolution of singularities we can write the morphism π as a composition of open embedding with normal crossing and a projective morphism. In these cases the statement is correct.

Important that we already have a morphism $i : \Omega(\pi_* E) \to \pi_*(\Omega(E))$ and we only have to show it is an isomorphism.

Step 2. On subcategory $D_{RS}(\mathcal{D}_X)$ the functor Ω commutes with $\pi_!$.

The same proof as in Step 1.

Step 3. The functor Ω commutes with functors $\pi^!$

It holds for smooth morphisms. Enough to check for a closed embedding $i : Z \to X$. Set $U = X \setminus Z$. Using standard exact triangle we reduce to commuting with j_*.

Step 4. The functor Ω commutes with duality.

$\Omega \pi_* \to \pi_* \Omega$

F clean: $\Omega \pi_*(F) \to \pi_* \Omega(F)$ is an

$i: Z \to X$, E on Z

$F = i_*(E)$

$i: Z \to X$ $\quad I = \pi \circ i: Z \to Y$

$\Omega \circ i_* = i_* \Omega$, $\Omega I_* = I_* \Omega$ on E

Enough to consider one MFE

$I: Z \to Y$

$I: Z \to \bar{Z} \to Y$

Ω commutes with $\pi^!$

Enough to consider

$i: Z \to X$ closed embed

$U = X \setminus Z$

$i_* i^! F \to F \to j_* j^! F$

$\Omega i_* i^! F \to \Omega F \to \Omega j_* j^! F$

$i_* i^! \Omega F \to \Omega F \to j_* j^! \Omega F$

$i^! \Omega = \Omega i^!$

$\exists$ a functorial morphism

$\nu: DR(\mathbb{D} F) \to \mathbb{D} DR(F)$

$\mathcal{O}_{X^{an}} \to \mathbb{D}(\omega_{an})$

$Id \to \mathbb{D}\mathbb{D}$

Thus we have established that on the category $D_{RS}(\mathcal{D}_X)$ the functor Ω commutes with all functors.

Step 5. Functor Ω maps $D_{RS}(\mathcal{D}_X)$ to $D_{con}(X)$

Step 6. Functor $\Omega : D_{RS}(\mathcal{D}_X) \to D_{con}(X)$ is fully faithful.

$\mathrm{Hom}(F, H) \to \mathrm{Hom}(\Omega F, \Omega H)$ is an isomorphism.

claim. $F, H \in D_{coh}(\mathcal{D}_X)$.

$\mathrm{Hom}_{D(\mathcal{D}_X)}(F, H) = H^0(\text{complexes on } pt)$.

claim. $\mathbb{D}F \overset{!}{\otimes} H \in D(\mathcal{D}_X)$

$\Delta : X \to X \times X$

$F \overset{!}{\otimes} H := \Delta^!(F \boxtimes H)$

$p : X \to pt$

$p_*(\mathbb{D}F \overset{!}{\otimes} H) \in D(pt)$

$= \mathrm{Hom}(\Phi_1(\mathcal{F}))$.

Step 7. Functor $\Omega : D_{RS}(\mathcal{D}_X) \to D_{con}(X)$ is essentially surjective.

Let X be smooth,
L - local system on X.

then $\exists$ a smooth RS D_X-module
E s.t.
~~Δ~~ $Loc(E) = L$

Proof. (Deligne).
E_{an} $X \subset \bar{X}$ good embedding
Extend E_{an} to coherent analytic $\mathcal{O}_{\bar{X}an}$ module
F_{an}.

$D \subset \mathbb{C}$ - disc.
$D^* = D - 0$
E smooth D_{an} - module on D^*
μ - monodromy operators
Find RS D_{an} - module M on D with the same μ and restricted exponents.

Then on D^* we have

isom tl bx $\simeq E$

t -parametel