D-modules-Lecture-9

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D-
modules-

9. Lecture 9. Continue the proof of RH correspondence

9.1. **GAGA.** Functor An.

Let X be a complex algebraic variety. We mostly assume that X is smooth, though this is not important.

We have described a functor $An : Coh(\mathcal{O}_X) \to Coh(O_{an})$. The properties of this functor are described by the following theorem

Theorem 9.2. *GAGA*

(i) Functor An is exact and hence lifts to derived category.

(ii) Functor An compatible with functor π^*

(iii) For a morphism $\pi : X \to Y$ we have a canonical morphism of functors $i : An \circ \pi_* \to \pi_{\cdot} \circ An$, both on categories $Coh(X)$ and on its derived category.

(iv) If morphism π is projective, then this morphism i is an isomorphism

(v) If X is a projective variety, then the functor An is an equivalence of categories.

In addition it is easy to see that the functor An maps differential morphisms between coherent sheaves into differential morphisms.

Using this fact it is easy to generalize GAGA to the category of coherent $\mathcal{D}_X$-modules.

Indication of the proof of GAGA.

This finishes the proof of RH.

Some technical details I will give as problems.

Main point in the proof of GAGA.

Let X be a projective variety. F is coherent $\mathcal{O}_{X^{an}}$-module. Then

1) $H^i(X^{an}, F)$ are f. dim.

2) $H^*(X^{an}, F \otimes \mathcal{O}_{P^n}(k)) = 0$ for large k, $i > 0$

Can assume $X = \mathbb{P}^N$

$\mathcal{U}$ - finite covering.

$\mathcal{V}$ - smaller covering.

$Ch(\mathcal{U}, F) \longrightarrow Ch(\mathcal{V}, F)$

$B' \subset B$ $\quad Res : \mathcal{O}_{an}(B) \longrightarrow \mathcal{O}_{an}(B')$ - is compact operator.

Corollary. $\mathcal{F}$-coherent $\mathcal{O}_{X_{an}}$ module.

$$0 \to \mathcal{K} \to \mathcal{O}(-k)^N \twoheadrightarrow \mathcal{F} \to 0$$

$\mathcal{F}(k)$ is generated by global section.

$$\mathcal{O}_{an}^N \to \mathcal{F}(k) \to 0$$

For any coherent $\mathcal{F}$ there exists a resolution

$$\cdots \to C_1 \to C_0 \to \mathcal{F} \to 0$$

$$C_i \simeq \mathcal{O}(a)^{k(i)}$$

$$\mathrm{Hom}_X(\mathcal{O}(i), \mathcal{O}(j)) = \mathrm{Hom}_{\mathcal{O}_{X_{an}}}(\mathcal{O}(i)_{an}, \mathcal{O}(j)_{an})$$

$$an: Coh(\mathcal{O}_X) \to Coh(\mathcal{O}_{an})$$

$$an: D_{coh}(\mathcal{O}_X) \to D_{coh}(\mathcal{O}_{an}).$$

$$H^i(\mathbb{P}, \mathcal{O}(k)) = H^i(\mathbb{P}_{an}, \mathcal{O}(k)_{an})$$

9.3. **Theorems de Finitude.** General method by Deligne. History.

Theorem 9.4. *Functor Ω maps holonomic complexes into constructible ones.*

Proof We will use the following

Criterion of constructibility. Let H be a complex of sheaves on X_{an}. It lies in $D_{con}(X)$ iff the following condition holds

(Con) For any locally closed subvariety $Z \subset X$ there exists an open dense smooth subvariety $U \subset Z$ such that the restriction $I_U^*(H)$ is locally constant in derived category.

Proof of the Theorem. Let F be a holonomic $\mathcal{D}_X$-complex, $H = \Omega(X)$. Since on some non-empty open subset $U \subset X$ the complex F is smooth we see that the restriction of H to U is LS and hence constructible.

Let U be the maximal open subset $U \subset X$ on which H is constructible. We want to show that $U = X$.

Suppose not.

Let W be an irreducible component of $X \setminus U$. I want to show that H is locally constant on some open dense subset $W_0 \subset W$. This, of course, will be a contradiction with maximality of U.

Claim. *I can reduce to the following situation.*

$X = \mathbb{P} \times W$, $W = p \times W$

U *and* $U \cup W$ *are open in* X.

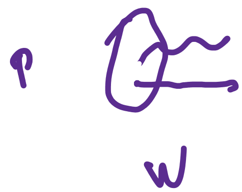
$W = X \setminus U$, $U \subset X \setminus W \subset X$ - open.

$F \in D(\mathcal{D}_X)_{h.e.}$ $H = \Omega(F)$ on X_{con}

$X_{an} = W_{an} + P_{an}$.

$P \times W$

N is a subspace of $X \setminus U$



$Z = X \setminus U$

$Z = W \cup T$ — disjoint union.

Assuming this consider the projection $p : X \to W$. This is a proper morphism, so $p_*(H) = p_*(\Omega(F)) = \Omega(p_*(F))$. since $p_*(F)$ is holonomic this complex is locally constant on some open subset $W_0 \subset W$.

We can assume that it is locally constant everywhere.

Let $Z = X \setminus U$ Then we have an exact triangle $F_U \to F \to F_Z$

Applying the functor p_* we get that $p_*(F_Z$ is constructible, and hence can assume locally constant.

However Z is a disjoint union of W and some other subset Z'. Hence $p_*(F_Z) = F_W \oplus R$ for some object R.

Thus we see that F_W is a direct summand of a locally constant object, and hence it is locally constant.

$X = U \cup T \cup W$

$Z = T \cup W$

H on X

$H_U \to H \to H_Z$

$j_{U*}(\mathcal{H}|_U) \to \mathcal{H} \to (i_Z)_* i_Z^* \mathcal{H}$

$p_*(\mathcal{H})$ is glued from
$p_*(\mathcal{H}_U)$ and $p_*(\mathcal{H}_Z)$.
$\Rightarrow$ $p_*(\mathcal{H}_Z)$ is constructible.
passing to open subset
$p_*(\mathcal{H}_Z)$ is loc. const.

$p_*\mathcal{H} = p_*\Omega(\mathcal{F}) = \Omega\, p_*(\mathcal{F})$:
Ω - holon. $\Rightarrow$
loc. const. on open subset,
$p_*(\mathcal{H}_Z)$ is locally constant

$Z = T \cup W$ - disjoint union.
$p_*(\mathcal{H}_Z) = p_*(\mathcal{H}_T) \oplus p_*(\mathcal{H}_W)$
$p_*(\mathcal{H}_Z)$ - loc. const. $\Rightarrow$
$p_*(\mathcal{H}_W)$ is loc. const.

$p_* \mathcal{H}_W = \mathcal{H}|_W$

$\mathcal{H}_W = (i_W)_* \mathcal{H}|_W$

$X = \mathbb{P} \times W$ $\qquad$ $i_W : W \to X$

$p\, i_W = Id$ $\qquad$ $\mathcal{H}_W = i_W^*(\mathcal{H})$

$p_*\, i_{W*} = Id.$

$p_* \mathcal{H}_W = p_* i_*(\mathcal{H}|_W)$

$p_*(\mathcal{H}_W) = \mathcal{H}|_W$

9.5. **Analytic corollary of RH.** Functor Sol.

F -coherent $\mathcal{D}_X$-complex. We define $Sol(F) := Rhom(F, O_{an})$.
Explicit description using resolutions.

Claim. $Sol(F) = \Omega(\mathbb{D}(F))$

Let $x \in X$. Set $O_{x,an}$ and $O_{x,for}$ be germs of analytic and formal lower series at the point x.

Proposition 9.5.1. . *let F be an RS $\mathcal{D}_X$ -complex. Then $Hom(F, O_{x,an}) = Hom(F, O_{x,for})$*

$X \supset U$ $\quad$ $Z = X \setminus U$

$Z = W \cup T$ $\quad$ W - compact curves of Z.

~~Exact~~

$W \subset X$ closed subvari...

Can assume

$W \subset X \subset \mathbb{A}^n$,

Can assume $X = \mathbb{A}^n$!

Lemma. $W \subset \mathbb{A}^n$, Zar...

We can consider as a projection

$W \to \mathbb{A}^k$ étale morphism

Extend this morphism to $\mathbb{A}^n$.

$\mathbb{A}^n \supset W \to \mathbb{A}^k$

$W \subset X =$ $\quad W \to X$, $X \to W$

$W \hookrightarrow W \times \mathbb{A}^k$