THE ALGEBRAIC THEORY OF D-MODULES

1. D-modules on affine spaces

1.1. **Philosophy and motivation.**

A system of linear equations is the set of vectors of coefficients. Some systems are equivalent, e.g.

$$\begin{cases} x+y=0 \\ x+2y+z=0 \end{cases} \approx \begin{cases} x+y=0 \\ y+z=0 \end{cases}$$

Systems are equivalent $\iff$ their coefficient vectors span the same linear subspaces. Thus:

system of linear equations $\longleftrightarrow$ vector space with a fixed basis

equivalence class of systems of linear equations $\longleftrightarrow$ vector space.

In modern linear algebra we just study vector spaces.

Solution of the system of linear equations = functional on the quotient by the vector space.

We will study systems of polynomial partial differential equations up to equivalence.

Consider first one variable x.
Polynomial differential equation = differential operator $d \in \mathcal{D}_1 = \mathbb{C}\langle x, \partial\rangle$,
where $\partial = \frac{d}{dx}$.
$\mathcal{D}_1$ is not commutative: $[x, \partial] = -1$ by Leibniz rule: $x\partial f - \partial(xf) = -f$.

Systems are equivalent if their equations generate the same left ideal. Thus
$\{$Systems of polynomial ODEs$\}/\sim$ cyclic modules over $\mathcal{D}_1$: (equations = relations).

Solutions of a system Ξ = morphisms of $\mathcal{D}_1$-modules $\Xi \to C^\infty(\mathbb{R})$.
Instead C^∞ we can consider $C^{-\infty}(\mathbb{R})$ - generalized functions.

We will study finitely generated $\mathcal{D}_n$-modules.

We will deduce properties:
$\mathcal{D}_n$-modules $\longrightarrow$ PDEs $\longrightarrow$ algebraic properties of solutions $\longrightarrow$ analytic properties of solutions.

Example: functional equation implies analytic continuation. E.g. $\Gamma(\lambda+1) = \lambda\Gamma(\lambda)$ allows to define $\Gamma(\lambda) := \lambda^{-1}\Gamma(\lambda+1)$, extending the area of definition of Γ. By induction, it gives meromorphic continuation to all of $\mathbb{C}$.

The function $|x|^\lambda$ is a smooth function of x for all complex λ with $\operatorname{Re}\lambda > -1$.
It satisfies:

$$\partial(|x|^\lambda) = \lambda|x|^{\lambda-1}$$

This enables to define $|x|^{\lambda-1}$ as $|x|^{\lambda-1} := \partial(|x|^\lambda)/\lambda$ for $\operatorname{Re}(\lambda-1) > -2$ unless $\lambda = 0$. We can continue this analytic continuation to the left, defining a meromorphic family of generalized functions.

In several variables: the same story, but the example becomes more complicated. We want to have meromorphic continuation of $|P(x_1,\ldots,x_n)|^\lambda$ for any polynomial P that grows to infinity in all directions.

This was an important open problem that initiated the theory of $\mathcal{D}_n$-modules.

1.2. **Definitions.** Fix an algebraically closed field $\mathbb{K}$ of characteristic 0.

Definition 1.1.

$$\mathcal{D}_n := \mathbb{K}\left\langle x_1,\ldots,x_n,\partial_1,\ldots,\partial_n\right\rangle / \left\langle [x_i,x_j]=0, [\partial_i,\partial_j]=0, [\partial_i,x_j]=\delta_{ij}\right\rangle.$$

Exercise 1.2. $\mathcal{D}_n$ is the subalgebra of $\operatorname{End}_{\mathbb{K}} \mathbb{K}[x_1,\ldots,x_n]$ generated by derivations and multiplication operators. By derivations we mean linear endomorphisms that satisfy the Leibnitz rule:

$$\partial(fg) = (\partial f)g + f(\partial g).$$

Exercise 1.3. Any $d \in \mathcal{D}_n$ can be written as

$$d = \sum c_{\alpha,\beta} x^\alpha \partial^\beta,$$

where α, β are multiindices, and $c_{\alpha,\beta} \in \mathbb{K}$. For example,

$$x_1\partial_1\partial_2 x_1 = x_1\partial_1 x_1\partial_2 = x_1^2\partial_1\partial_2 - x_1[x_1,\partial_1]\partial_2 = x_1^2\partial_1\partial_2 + x_1\partial_2$$

Definition 1.4. Let $\mathbb{K} = \mathbb{C}$ and let M be a finitely generated $\mathcal{D}_n$-module.
Then a solution of M is a homomorphism from M to some $\mathcal{D}_n$-module of functions (say, $C^\infty(\mathbb{R}^n)$ or $C^{-\infty}(\mathbb{R}^n)$).

Example 1.5. To any linear system of polynomial PDEs $\{L_1 f = 0, \ldots, L_k f = 0\}$, we associate the $\mathcal{D}_n$-module $\mathcal{D}_n/\langle L_1,\ldots,L_k\rangle$.

Exercise 1.6. The center is: $Z(\mathcal{D}_n) = \mathbb{K}$.

Notation 1.7. $\mathcal{M}$ is the category of left $\mathcal{D}_n$-modules, $\mathcal{M}^r$ is that of right modules, $\mathcal{M}^f$ — finitely generated left modules.

1.3. **Dimension.**

Lemma 1.8. *For any $M \in \mathcal{M}(D_n)$ either $M = 0$ or $\dim_{\mathbb{K}} M = \infty$.*

Proof. If $\dim M < \infty$ then $0 = \operatorname{tr}[\partial_1, x_1] = \operatorname{tr} 1 = \dim_{\mathbb{K}} M$. □

This motivates other ways of measuring the "size" of a module.

Definition 1.9. A filtered algebra is an algebra A equipped with an increasing sequence of subspaces $F^iA, i \geq 0$, $F^iA \subset F^{i+1}A$, $\bigcup_i F^iA = A$,
such that $1 \in F^0A$ and $F^iA \cdot F^jA \subset F^{i+j}A$.

A filtration is called good if F^iA is f.g. over F^0A, and $F^{i+1}A = F^1A \cdot F^iA$ for $i \gg 0$ (i large enough).

Example 1.10. $A = \mathbb{K}[y_1, \ldots, y_m]$, $F^iA := \{\deg \leq i\}$.

Example 1.11. Bernstein filtration: $F^i\mathcal{D}_n := \operatorname{span}\left\{x^\alpha\partial^\beta \;\middle|\; |\alpha| + |\beta| \leq i\right\}$

Definition 1.12. For a filtered algebra (F^iA) the associated graded algebra is

$$\operatorname{Gr}_F A := \bigoplus_i \left(F^iA/F^{i-1}A\right). \quad F^{-1}A := 0.$$

Example 1.13. $\operatorname{Gr}\mathcal{D}_n = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_n]$.

Let A be a good filtered algebra.

Definition 1.14. A filtered A-module M is a module equipped with an increasing sequence of subspaces F^iM, such that

$$F^iA \cdot F^jM \subset F^{i+j}M, \quad \bigcup_i F^iM = M.$$

A filtration is called good if:
F^iM are finitely generated over F^0A, and $F^{i+1}M = F^1A \cdot F^iM$ for $i \gg 0$.

Exercise 1.15. Any two good filtrations are comparable, i.e. there exists m s.t.

$$F^{i-m}M \subset \Phi^iM \subset F^{i+m}M \quad \forall i$$

Proof. Suppose that $\forall i : F^{N+i}M = (F^1A)^i\, F^NM$.
Since F^NM is finitely generated over F^0A, and $\bigcup_i \Phi^iM = M$, one can assume that $F^NM \subset \Phi^{N'}M$. Then

$$F^{N+i}M = \left(F^1A\right)^i F^NM \subset \left(F^1A\right)^i \Phi^{N'}M \subset \Phi^{N'+i}M.$$

By the same argument, $\Phi^{N'+i}M \subset F^{N''+i}M$. Now

$$F^{N+i}M \subset \Phi^{N'+i}M \subset F^{N''+i}M$$

for all $i \geq 0$. □

Remark 1.16. A filtered algebra is good $\Leftrightarrow$ it is good as a module over itself.

We define $\operatorname{Gr}_F M$ for a filtered module in a similar way as for algebras.
We will sometimes write A^i for F^iA and M^i for F^iM if the filtration is understood.

Exercise 1.17. Assume that A^i is a good filtered algebra. Then

$$F^iM \text{ is good} \Leftrightarrow \operatorname{Gr}_F M \text{ is finitely generated over } \operatorname{Gr}_F A.$$

Proof. Suppose that $M^i := F^iM$ is good, and $M^{i+1} = A^1M^i$ starting from some N. Take generators m_i of M^N over A^0. Then they their symbols generate $\operatorname{Gr} M$ over $\operatorname{Gr} A$. Conversely, suppose that $\operatorname{Gr} M$ is finitely generated over $\operatorname{Gr} A$ by elements m_i, $\deg m_i = d_i$. Then the filtras M^i are obtained by iterated extensions from M^j/M^{j-1}, which are finitely generated over A^0, so M^i are also finitely generated over A^0. On the other hand,

$$A^1M^i/M^i \simeq \left(A^1/A^0\right)\left(M^i/M^{i-1}\right)$$

and if A is good then

$$M^{i+1}/M^i = \sum_j \left(A^{i-j+1}/A^{i-j}\right)\left(M^j/M^{j-1}\right) =$$

$$\left(A^1/A^0\right)\sum_j \left(A^{i-j}/A^{i-j-1}\right)\left(M^j/M^{j-1}\right) = \left(A^1/A^0\right)\left(M^i/M^{i-1}\right)$$

where j runs through the degrees of the generators of $\operatorname{Gr} M$ over $\operatorname{Gr} A$ and i is assumed to be larger than the maximum of these degrees. $\square$

Exercise 1.18. A module M over a good filtered algebra A admits a good filtration if and only if M is finitely generated.

Proof. Suppose first that M admits a good filtration. Then $M^{i+1} = A^1M^i$ for $i \geq N$ and $M^N$is finitely generated over A^0. Thus the generators of M^Nover A^0 generate M.

Conversely, assume M is generated by a finite set $x_i{}_{i=1}^k$ and consider the filtration $M^i := A^ix_1 + \cdots + A^ix_k$. Since A^i is finitely generated over A^0, M^i is also finitely generated over A^0, and $M^{i+1} = A^1M^i$ as long as $A^{i+1} = A^1A^i$.

Since any element $m \in M$ is representable as $\sum_{i=1}^k a_ix_i$, we have $m \in M^n$, where n is big enough so that $a_i \in A^n$. Therefore $M = \bigcup_i M^i$. $\square$

Definition 1.19. For a filtered module F^iM and a short exact sequence

$$0 \to L \to M \to N \to 0,$$

define induced filtrations on L and M by

$$F^iL := F^iM \cap L \text{ and } F^iN := F^iM/F^iL.$$

A map $f : M \to R$ of filtered modules is called **strict** if $f\left(M^i\right) = f\left(M\right) \cap R^i$.

Exercise 1.20. Let $L \to M \to N$ be an exact sequence of filtered modules and strict maps between them. Then the corresponding sequence $\operatorname{Gr} L \to \operatorname{Gr} M \to \operatorname{Gr} N$ is also exact.

Exercise 1.21. For a good filtered module F^iM and a short exact sequence

$$0 \to L \to M \to N \to 0,$$

the induced filtration on N is good, and if $\operatorname{Gr} A$ is Noetherian then so is the induced filtration on L.

Proof. Note that all maps in $0 \to L \to M \to N \to 0$ are strict. Thus $\operatorname{Gr} N$ is a factor of $\operatorname{Gr} M$, so it's finitely generated over $\operatorname{Gr} A$; and $\operatorname{Gr} L$ is a submodule of $\operatorname{Gr} M$, so if $\operatorname{Gr} A$ is Noetherian then $\operatorname{Gr} L$ is finitely generated over it. □

Remark 1.22. The category of filtered modules is not Abelian — say, the shift map doesn't have a cokernel.

Theorem 1.23. *Suppose that A is a good filtered algebra and $\operatorname{Gr}_F A$ is Noetherian. Then A is also Noetherian.*

Proof. Let M be a finitely generated A-module, and $L \subset M$. Pick a good filtration F. Then $\operatorname{Gr}_F M$ is finitely generated, $\operatorname{Gr}_F A$ is Noetherian, hence $\operatorname{Gr}_F L$ is finitely generated, so F^iL is good, so L is finitely generated over A. □

Corollary 1.24. *$\mathcal{D}_n$ is Noetherian.*
The universal enveloping algebra of any finite dimensional Lie algebra is Noetherian.

Notation 1.25.
(i) If $f, g : \mathbb{N} \to \mathbb{Z}$ we say $f \sim g$ if $f = g$ for $i \gg 0$.
(ii) $\Delta f(i) := f(i+1) - f(i)$.

Exercise 1.26. Let $f : \mathbb{N} \to \mathbb{Z}$ be an integer sequence. Show that the following are equivalent:
(1) f is eventually polynomial of degree $\leq d$.
(2) Δf is eventually polynomial of degree $\leq d-1$.
(3) $f(j) \sim \sum_{i=0}^{d} e_i \binom{j}{i}$ where $e_i \in \mathbb{Z}$.

Theorem 1.27 (Hilbert)**.** *Let $R = \bigoplus R^i$ be a graded finitely generated $\mathbb{K}[x_1, \ldots, x_n]$-module. Then $b(i) := \dim_{\mathbb{K}} R^i$ is eventually polynomial of degree $d \leq n-1$.*
The number $d+1$ is called the functional dimension of R.

Proof. Define $(R[1])^i := R^{i+1}$.

$$0 \to \ker x_n \to R \xrightarrow{x_n} R[1] \to \operatorname{coker} x_n \to 0$$

This is a morphism of graded modules, so $\ker x_n$ and $\operatorname{coker} x_n$ are graded modules. Thus:

$$\dim_{\mathbb{K}} (\ker x_n)^i - \dim_{\mathbb{K}} R^i + \dim_{\mathbb{K}} R^{i+1} - \dim_{\mathbb{K}} (\operatorname{coker} x_n)^i = 0$$

On the other hand, x_n acts by 0 on both ker and coker, so by induction on n we know that $\Delta \dim_{\mathbb{K}} R^i$ is eventually polynomial, therefore so is $\dim_{\mathbb{K}} R^i$. □

Corollary 1.28. *Let F^iM be a good filtered $\mathcal{D}_n$-module.*
Then $b_M(i) := \dim F^iM$ is eventually polynomial of degree $\leq 2n$.

Proof. $b_M = b_{\operatorname{Gr} M}$, and note that $\operatorname{Gr} \mathcal{D}_n \simeq \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. □

Remark 1.29. Now since for any two filtrations $F^{i-k}M \subset \Phi^i M \subset F^{i+k}M$, the degree and leading coefficient are invariant (for a fixed filtration of the algebra).

Definition 1.30. The degree of $b_M(i)$ is called the dimension of M and denoted $d(M)$, and the leading coefficient of $d(M)!b_M(i)$ is called the (Bernstein) degree of M and denoted $e(M)$.

2. Bernstein inequality

Theorem 2.1 (Bernstein inequality)**.** *Let M be a finitely generated $\mathcal{D}_n$-module. If $M \neq 0$ then*

$$n \leq d(M) \leq 2n.$$

Remark 2.2. Note that $d(M) \neq 0$ is equivalent to $\dim_{\mathbb{K}} M = \infty$, so Bernstein's inequality can be viewed as a generalization of that.

Example 2.3.

(i) $d(\mathcal{D}_n) = 2n$.
(ii) $d(P_n) = n$, where P_n is the module of polynomials.
(iii) Let δ denote the Dirac's δ-function at zero, and Δ denote the $\mathcal{D}_n$-module generated by it.
In other words, $\Delta = \mathcal{D}_n/\langle x_1, \ldots, x_n\rangle$. Then $d(\Delta) = n$.

2.1. **Proof of Bernstein's inequality.**

Let $M \in \mathcal{M}^f(\mathcal{D}_n)$ and define

$$N^0 := \ker(x_n|_M) \subset M, \quad N^\ell := \partial_n^\ell N^0.$$

They are viewed as $\mathcal{D}_{n-1}$-modules.

Lemma 2.4. $\partial_n^\ell : N^0 \simeq N^\ell$ *and* N^ℓ *are linearly independent.*

Proof. Let $m \in N^0$, $m_\ell := \partial_n^\ell m$. Then since $\left[x_n, \partial_n^\ell\right] = -\ell\partial_n^{\ell-1}$, we have

$$x_n m_\ell = -\ell m_{\ell-1}$$

Thus x_n "inverts" ∂_n on N^ℓ up to a scalar. Thus $\partial_n^\ell : N^0 \simeq N^\ell$.

N^ℓ are linearly independent because they are different eigenspaces of $x_n\partial_n$. □

Corollary 2.5. *If* $\ker x_n|_M \neq 0$ *and Bernstein's inequality holds for* $\mathcal{D}_{n-1}$ *then*

$$d(M) \geq n.$$

We use notation $\mathcal{D}_n^\ell := F^\ell \mathcal{D}_n$.

Proof. Let $m \neq 0$, $m \in \ker(x_n|_M)$. Then

$$\mathcal{D}_n^{2\ell} m \supset \bigoplus_{i=0}^{\ell} \partial_n^i \mathcal{D}_{n-1}^\ell m = \bigoplus_{i=0}^{\ell} \mathcal{D}_{n-1}^\ell \partial_n^i m.$$

Thus

$$\dim \mathcal{D}_n^{2\ell} m \geq \ell \dim \mathcal{D}_{n-1}^\ell m \geq \text{const} \cdot \ell^n.$$

□

Corollary 2.6. *If* $\operatorname{coker} x_n \neq 0$ *and Bernstein's inequality holds for* $\mathcal{D}_{n-1}$ *then*

$$d\,(M) \geq n.$$

Proof. By the previous corollary we can assume $\ker x_n = 0$. Now,

$$\operatorname{coker} x_n = M/x_n M$$

is a $\mathcal{D}_{n-1}$-module. Assume it's finitely generated. Then

$$\dim F^i M - \dim x_n F^{i-1} M \geq c i^{n-1}.$$

If it's not finitely generated then take a finitely generated submodule. Thus

$$\Delta \dim F^i M \geq c i^{n-1},$$

so $d\,(M) \geq n$. □

Exercise 2.7 (Amitsur-Kaplansky lemma)**.** Let $\mathbb{L}$ be an uncountable algebraically closed field. Let V be an $\mathbb{L}$-vector space of countable dimension.

Then any linear operator on $T : V \to V$ has nonempty spectrum,
i.e. $T - \lambda \operatorname{Id}$ is not invertible for some $\lambda \in \mathbb{L}$.

Proof. Assume by way of contradiction that the spectrum of T is empty.

Let $v \in V$ be a non-zero vector. Then

$$\{(T-\lambda)^{-1} v,\ \lambda \in \mathbb{L}\}$$

is an uncountable set. Thus it is linearly dependent. Picking a dependence and bringing it to a common denominator we obtain $p(T)v = 0$, for some polynomial p.

On the other hand, p is a product of linear factors, thus $p(T)$ is invertible and has no kernel. Contradiction. □

Proof of Bernstein inequality. Extend the field so that it becomes uncountable.
By the previous lemma, $x_n - \lambda$ is not invertible for some λ. Apply the automorphism $x_n \mapsto x_n - \lambda$. Now the theorem follows by induction from the previous corollaries. □

2.2. **Joseph's proof of Bernstein inequality.**
Idea: $\mathcal{D}_n$ has no two-sided ideals, thus its modules have no annihilators.
Thus, for any module M, $\mathcal{D}_n$ embeds into $\operatorname{End}_K(M)$. Since

$$\dim \mathcal{D}_n^i \sim (i)^{2n} \text{ and } \dim_K \operatorname{End}(F^i M) = \dim(F^i(M))^2 \sim i^{2d(M)},$$

we obtain that $d(M) \geq n$.

Lemma 2.8. *(Exc). The center of* $\mathcal{D}_n$ *is* $\mathbb{K}$.

Lemma 2.9. *Let* $F^i M$ *be a good filtration on a* $\mathcal{D}_n$*-module* M*. Then the action defines an embedding*

$$\mathcal{D}_n^i \hookrightarrow \operatorname{Hom}_k(F^i M, F^{2i} M).$$

Proof. The map is defined by definition of filtration. Let us prove that it is an embedding by induction on i. For $i = 0$ this is obvious. For a bigger i, let $d \neq 0$ lie in the kernel. Since d is not scalar and thus not central, there exists l such that

$$[d, x_l] \neq 0 \text{ or } [d, \partial_l] \neq 0.$$

Assume WLOG $[d, x_1] \neq 0$. Then $[d, x_1] \in \mathcal{D}_n^{i-1}$and by the induction hypothesis $[d, x_1]v \neq 0$ for some $v \in F^{i-1}M$. However,

$$[d, x_1]v = dx_1v - x_1dv = 0, \text{ since } v, x_1v \in F^iM.$$

We arrived at a contradiction and thus $d = 0$. □

Joseph's Proof of Bernstein inequality.
Suppose by way of contradiction that $d(M) \leq n - 1$. Then

$$\dim Hom_k(F^iM, F^{2i}M) < ci^{n-1}(2i)^{n-1} = c' i^{2n-2}.$$

On the other hand $\dim \mathcal{D}_n^i > c'' i^{2n}$. This contradicts the previous lemma. □

This idea also allows to prove a similar theorem for Gelfand-Kirillov dimension of modules over algebraic Lie algebras.

Later we will state without proof a deep geometric theorem that implies the Bernstein inequality.

3. Holonomic modules, and an application.

In the next section we will state without proof a deep geometric theorem that implies the Bernstein inequality.

Definition 3.1. A finitely generated $\mathcal{D}_n$-module M is called **holonomic** if $d(M) = n$.

Exercise 3.2. If M is holonomic then it has length at most $e(M)$.

Corollary 3.3. *(Exc). Let M be a $\mathcal{D}_n$-module, and let F^iM be a (not necessary good) filtration on M. Suppose that* $\dim F^iM \leq e\binom{i}{n}$ *for some e. Then M is finitely-generated. Moreover, it is holonomic and of length at most e.*

We are now ready to give the first application to the theory of distributions. Let P be a polynomial in n real variables. Let $\lambda \in \mathbb{C}$ with $Re\lambda > -1$ and consider the locally integrable function $|P|^\lambda$.

Theorem 3.4. *(Bernstein, Gelfand, Gelfand, Atiya, ...) Consider $|P|^\lambda$ as a family of generalized functions. Then this family has meromorphic continuation to the entire complex plane with poles in a finite number of arithmetic progressions.*

This theorem follows from an algebraic statement saying that there exists a differential operator d with polynomial coefficients (that depend also on λ), and a polynomial b in λ such that $d|P|^\lambda = b(\lambda)|P|^{\lambda-1}$. Let us formulate this algebraic statement more precisely, over any field, and prove it.

Notation 3.5. Fix a polynomial $P \in k[x_1, \ldots, x_n]$. Let $K := k(\lambda)$ be the field of rational functions. Consider the $\mathcal{D}_n(K)$-module

$$M_P := M'_p \otimes_{k[\lambda]} K, \text{ where } M'_p := span\{QP^{\lambda-l}\},$$

where $Q \in k[x_{1,} \ldots, x_{n,}\lambda]$ and $l \in \mathbb{Z}_{\geq 0}$, with the relations $PP^{\lambda-l} = P^{\lambda-l+1}$, and the action of $\mathcal{D}_n[\lambda]$ given by

$$\partial_i(QP^{\lambda-l}) = \partial_i(Q)P^{\lambda-l} + Q(\lambda - l)\partial_i(P)P^{\lambda-l-1}.$$

Lemma 3.6. *The module M_P is finitely generated, and, moreover, holonomic.*

Proof. Define a filtration on M_P by

$$F^i M_P := \{QP^{\lambda-i} \ s.t. \ \deg Q \leq (\deg P+1)i\}.$$

It satisfies $\dim F^i M_P \leq ci^n$. It's not clear whether this is a good filtration, but by the Corollary above we still get that $M_P$is finitely generated and holonomic. □

Corollary 3.7. *There exist $d \in k[x_1,\ldots,x_n,\partial_{1,}\ldots,\partial_{n,}\lambda]$ and $b \in k[\lambda]$ s.t.*

$$dP^\lambda = bP^{\lambda-1}.$$

Proof. Consider the increasing chain of submodules

$$\mathcal{D}_n(K)P^\lambda \subset \mathcal{D}_n(K)P^{\lambda-1} \subset \ldots.$$

This chain has to stabilize. Thus $\tilde{d}P^{\lambda-k} = P^{\lambda-k-1}$ for some $\tilde{d} \in \mathcal{D}_n(K)$. Applying the automorphism $\lambda \mapsto \lambda+k$ we get that $\hat{d}P^\lambda = P^{\lambda-1}$ for some $\hat{d} \in \mathcal{D}_n(K)$. Now, we can decompose $\hat{d} = \frac{d}{b}$. □

Finally, let us show that holonomic modules are cyclic.

Theorem 3.8. *Let R be a simple Noetherian non-Artinian ring, and M a finitely generated Artinian left R-module. Then M is cyclic.*

Proof. By induction on length, we assume that $M = R\langle u,v\rangle$ with Rv simple. Since R is not Artinian, and M is Artinian, there exists d such that $du=0$. On the other hand, since R is simple, $R = RdR$, so there is b such that $dbv \neq 0$.

Let us show that $M = R\langle u+bv\rangle$. Indeed, $d(u+bv) = dbv \in Rv$, so since Rv is simple, $R\langle u+bv\rangle \supset Rv$. Thus $v, bv \in \langle u+bv\rangle$, and thus also $u \in \langle u+bv\rangle$. Since $M = R\langle u,v\rangle, M = R\langle u+bv\rangle$. □

Corollary 3.9. *Holonomic $\mathcal{D}_n$-modules are cyclic.*

4. Associated varieties and singular support

Let A be a finitely-generated commutative $\mathbb{K}$-algebra without nilpotents.

Definition 4.1. Let M be an A-module. Denote by $\operatorname{Ann} M$ the annihilator ideal $AnnM := \{a \in A \,|aM = 0\}$ and define the support $\operatorname{Supp}M$ to be the zeros of $AnnM$ in the maximal spectrum $\operatorname{Specm} A$.

If M is finitely generated then $\operatorname{Supp}M$ is the support of the coherent sheaf on $\operatorname{Specm} A$ that corresponds to M. This follows from Nakayama's lemma. If $A = \mathbb{K}[x_1,\ldots,x_n]$ then $\operatorname{Specm} A = \mathbb{A}^n$.

The algebra $\mathcal{D}_n$ is not commutative, and in order to associate a variety to a finitely-generated $\mathcal{D}_n$ module we will use the associated graded algebra $\mathbb{K}[x_1,\ldots,x_n,\xi_1,\ldots,\xi_n]$

Definition 4.2. Two modules M, N over the same algebra are called Jordan-Holder equivalent if there exist two increasing chains of the same finite length of submodules $0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$ and $0 = N_0 \subset N_1 \subset \cdots \subset N_m = N$ and a permutation $\sigma \in Sym_m$ s.t. $M_i/M_{i-1} \simeq N_{\sigma(i)}/N_{\sigma(i)-1}$ for any i.

Lemma 4.3. *Let F, Φ be two good filtrations on a $\mathcal{D}_n$-module M. Then $Gr_F M$ and $Gr_\Phi M$ are Jordan-Holder equivalent.*

Proof. Case 1. F, Φ are neighbors, i.e. $F^iM \subset \Phi^iM \subset F^{i+1}M \subset \Phi^{i+1}M$. In this case we have a well-defined map $\phi : Gr_F M \to Gr_\Phi M$, and $Ker\phi \simeq CoKer\phi$. Thus $Gr_F M$ and $Gr_\Phi M$ are Jordan-Holder equivalent.

In the general case, one can construct a sequence of neighboring filtrations $F^iM + \Phi^{i+l}M$, which starts with F and ends with a shift of Φ. □

Lemma 4.4. *Let $0 \to L \to M \to N \to 0$ be a short exact sequence of A-modules. Then* $\mathrm{Supp}M = \mathrm{Supp}N \cup \mathrm{Supp}L$.

Proof. Clearly $\mathrm{Ann}\, M \subset \mathrm{Ann}\, N \cap \mathrm{Ann}\, L$. Now, if $a \in \mathrm{Ann}\, N \cap \mathrm{Ann}\, L$ then for any $m \in M$ we have $am \in L$ and thus $a^2m = 0$. This shows that

$$\mathrm{Ann}\, N \cap \mathrm{Ann}\, L \subset \mathrm{Rad}\,\mathrm{Ann}\, M.$$

So $\mathrm{Ann}\, M \subset \mathrm{Ann}\, N \cap \mathrm{Ann}\, L \subset \mathrm{Rad}\,\mathrm{Ann}\, M$ and thus their zero sets coincide. □

Corollary 4.5. *If two A-modules are Jordan-Holder equivalent then they have the same support.*

Definition 4.6. The associated variety $AV(M)$ of a finitely-generated $\mathcal{D}_n$-module M is the support of $Gr_F M$ for some good filtration F.

By definition, $AV(M)$ is a closed subset of the affine space $\mathbb{A}^{2n}$. By Lemma 4.3 and Corollary 4.5 it does not depend on the choice of a good filtration.

Lemma 4.7 (Bernstein)**.** *Let M be a $\mathcal{D}_n$-module generated by a finite subset $S \subset M$. Let $I \subseteq \mathcal{D}_n$ be the annihilator of S, and let $J \subseteq A := \mathbb{K}[x_1, ..., x_n, \xi_1, ..., \xi_n]$ be the ideal generated by the symbols of the elements of I. Then the associated variety $AV(M)$ is the zero set of J.*

Proof. We first show that J vanishes on $AV(M)$. Let $S = \{m_1, \dots, m_s\}$. Define a good filtration on M by $F_i(M) = \mathcal{B}_i m_1 + ...\mathcal{B}_i m_s$, where $\mathcal{B}_i \subset \mathcal{D}_n$ is the $i-$th Bernstein filtration. If $d \in I \cap \mathcal{B}_j$ satisfies $dm_j = 0$ for any $m_i \in S$, then for any $c \in \mathcal{B}_i$ we have

$$dcm_j = [d, c]m_l + cdm_l = [d, c]m_l \in F_{i+j-1}M.$$

Thus, $\sigma(d)\widetilde{m_l} = 0$ where $\sigma : \mathcal{D}_n \longrightarrow A$ is the symbol map, and $\widetilde{m_l}$ is the image of m_l in $gr_F M$. Since $\{\widetilde{m_l}\}_{l=1}^s$ generate $\mathrm{Gr}_F\, M$, we get that $\sigma(d) \subset \mathrm{Ann}(\mathrm{Gr}_F(M))$. Thus $J \subset Ann(\mathrm{Gr}_F(M))$ and thus J vanishes on $AV(M)$.

Let us now show by induction on s that $Ann(\mathrm{Gr}_F(M)) \subset Rad(J)$. It is enough to show that for any homogeneous polynomial $a \in Ann(\mathrm{Gr}_F(M))$, there exist a natural number t and an operator $d \in I$ such that $\sigma(d) = a^t$.

For $s = 1$ note that by definition of $\mathrm{Gr}_F\, M$, there exist operators $c, c' \in \mathcal{D}_n$ such that $c \in \mathcal{B}_{deg(a)}, c' \in \mathcal{B}_{deg(a)-1}$, $\sigma(c) = a$, and $cm_1 = c'm_1$. Then $d := c - c' \in I$ and $\sigma(d) = a$.

For the induction step, we will repeatedly use the fact that for any submodule $L \subset M$, we have $AV(L) \subset AV(M)$. This is so since $\mathrm{Gr}_{F'}\, L \subset \mathrm{Gr}_F(M)$, where F' is the induced filtration on L. Note also that a vanishes on $AV(M)$.

Let $S_1 := \{m_1, \dots, m_{s-1}\}$, let $I_1 \subset \mathcal{D}_n$ denote its annihilator, and L_1 denote the submodule of M generated by S_1. Since $AV(L_1) \subset AV(M)$, a vanishes on $AV(L_1)$ and thus the induction hypothesis implies that there exist $d_1 \in I_1$ and a power t_1 such that $\sigma(d_1) = a^{t_1}$. Let $S_2 := \{d_1 m_s\}$ and L_2 be the submodule of M generated by it. Since $AV(L_2) \subset AV(M)$, a vanishes on $AV(L_2)$ and thus the base of the induction implies that there exist $d_2 \in \mathcal{D}_n$ and a power t_2 such that $d_2 d_1 m_s = 0$ and $\sigma(d_2) = a^{t_2}$. Now take $d := d_2 d_1$ and $t := t_1 + t_2$. □

Now we would like to argue that the dimension of $AV(M)$ equals $d(M)$. This follows from Hilbert's definition of dimension.

Definition 4.8. Let $X \subset \mathbb{A}^n$ be an affine algebraic variety and let $I \subset A := \mathbb{K}[x_1, \dots, x_n]$ be the ideal of functions vanishing on X. The standard filtration on A induces a good filtration F^i on A/I. By Theorem 1.27, the function $f(i) := \dim F^i(A/I)$ is eventually polynomial. Define $\dim X$ to be the degree of this polynomial.

Exercise 4.9. For any $M \in \mathcal{M}^f(\mathcal{D}_n)$, $\dim AV(M) = d(M)$.

4.1. **Digression on several definitions of dimension of algebraic varieties.** Let us first define dimension by properties and then discuss several equivalent definitions.

Definition 4.10. A dimension is a correspondence of a non-negative integer to every algebraic variety such that

(i) $\dim(\mathbb{A}^n) = n$
(ii) For a (locally closed) subvariety $Y \subset X$, $\dim(X) = \max(\dim Y, \dim(X \setminus Y))$.
(iii) For a finite epimorphism $\varphi : X \to Y$, $\dim X = \dim Y$.

The uniqueness of dimension follows from the Noether normalization lemma.

Lemma 4.11. *For any affine algebraic variety X, there exists a finite epimorphism $\varphi : X \to \mathbb{A}^n$ for some n.*

A finite morphism is a morphism $\varphi : X \to Y$ such that for any open affine $U \subset Y$, the preimage $\varphi^{-1}(U)$ is affine and the algebra $\mathcal{O}(\varphi^{-1}(U))$ of regular functions on it is finitely-generated as a module over $\mathcal{O}(U)$. Finite morphisms are proper and have finite fibers.

There are several constructions of the dimension function. One of them is the Krull dimension: the maximal length of a strictly increasing chain of closed irreducible non-empty subsets, minus one. Another is the Hilbert dimension: define the dimension of a variety as the maximal among the dimension of open affine subvarieties, and for an affine subvariety use Definition 4.8. Another way is to define the dimension of an affine variety to be the transcendence degree of its field of rational functions (over $\mathbb{K}$).

4.2. **The geometric filtration.** There is another very natural filtration on the algebra $\mathcal{D}_n$ - filtration by the degree of the differential operator. In other words, $\deg x_i = 0$, $\deg \partial_i = 1$. This filtration is called *the geometric filtration*.

Note that the associated graded algebra by this filtration is again isomorphic to $\mathbb{K}[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$, but with a different grading. Note also that this is a good algebra filtration, and all the lemmas we proved about the arithmetic filtration hold for the geometric filtration, with one exception: the geometric filtras are infinite dimensional. Thus we cannot define a "geometric dimension", but we can define a "geometric associated variety". It is called the singular support, or the characteristic variety.

Definition 4.12. Let $M \in \mathcal{M}^f(\mathcal{D}_n)$ and let F be a filtration on M which is good with respect to the geometric filtration on $\mathcal{D}_n$. Define the singular support of M to be

$$\mathrm{SingSupp}(M) := \mathrm{Supp}(\mathrm{Gr}_F M).$$

Proposition 4.13. $d(M) = \dim \mathrm{SingSupp}(M)$.

We will now sketch an elementary proof, and give a deeper proof in the next section.

Sketch of proof. It is enough to prove the proposition for a cyclic module $M = \mathcal{D}_n/I$. Consider a sequence of filtrations F_l^i on $\mathcal{D}_n$ given by $\deg_l(x_i) = 1$, $\deg_l(\partial_i) = l$. Then for any $d \in I$ and for l big enough, the symbol of d with respect to F_l is the highest homogeneous summand of the symbol of d with respect to the geometric filtration. Thus it is enough to show that $\dim \mathrm{Supp}\, \mathrm{Gr}_{F_l} M = \dim \mathrm{Supp}\, \mathrm{Gr}_{F_{l+1}} M$ for every l, where $F_i^l M = F_i^l(\mathcal{D}_n)/(I \cap F_i^l(\mathcal{D}_n))$. By Hilbert's definition of dimension this amounts to computing that the eventual-polynomial functions $\dim F_i^l M$ and $F_i^{l+1} M$ have the same degrees.

Warning: the filtrations F_l on $\mathcal{D}_n$ are not good by our definition. However, they are still "almost" good, namely the Rees algebra $\bigoplus_{i \in \mathbb{Z}} t^i F_l^i \mathcal{D}_n$ is finitely generated. It is possible to work with such filtrations in a similar way to good filtrations. □

4.3. **Involutivity of the associated variety.** The affine space $\mathbb{A}^{2n}$ has a natural symplectic form. On the tangent space at zero it is given by

$$\omega(x_i, x_j) = \omega(y_i, y_j) = 0, \quad \omega(x_i, y_j) = \delta_{ij}.$$

Extending this formula by Leibnitz rule we get the Poisson brackets on the whole algebra $k[x_1, \dots, x_n, y_1, \dots, y_n]$. In fact, these Poisson brackets can be obtained from $\mathcal{D}_n$: for any two homogeneous polynomials $a, b \in k[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$ choose differential operators $c, d \in \mathcal{D}_n$ with symbols a, b. Then, $\{a, b\}$ is the symbol of $[a, b]$. Another way to obtain this form is to identify $\mathbb{A}^{2n}$ with the cotangent bundle $T^*\mathbb{A}^n$.

Definition 4.14. An algebraic subvariety X of $\mathbb{A}^{2n}$ is called *coisotropic* or *involutive* or *integrable* if the ideal of polynomials that vanish on X is stable under the Poisson brackets.

Remark 4.15. This is equivalent to saying that the tangent space to X at every smooth point includes its orthogonal complement inside the tangent space to $\mathbb{A}^{2n}$ w.r. to the symplectic form.

Theorem 4.16 (Gabber, Kashiwara-Kawai-Sato)**.**
For any $M \in \mathcal{M}^f(\mathcal{D}_n)$, both $AV(M)$ and $\mathrm{SingSupp} M$ *are coisotropic.*

Note that any coisotrpoic subvariety has dimension at least n, and thus this theorem implies the Bernstein inequality.

The proof of this theorem is outside the scope of our course. It is not difficult in fact to show that $AnnGr(M)$ is closed under the Poisson brackets. The difficulty is to show that so does its radical. This theorem has applications to the theory of invariant distributions, in addition to the ones that Bernstein's inequality does.

Since $\mathrm{SingSupp} M$ is (almost by definition) invariant under homotheties in $\xi_1, \dots, \xi_n$, Theorem 4.16 implies the following corollary.

Corollary 4.17. *For any holonomic $M \in \mathcal{M}^f(\mathcal{D}_n)$, $\mathrm{SingSupp} M$ is a finite union of conormal bundles to closed subvarieties of $\mathbb{A}^{2n}$.*

We will not use the theorem and the corollary, this was just to give a geometric intuition.

4.4. **Irreducible non-holonomic $\mathcal{D}_n$-modules.** We will now show that there are many irreducible non-holonomic $\mathcal{D}_n$-modules.

Definition 4.18. We call a coisotropic homogeneous closed subvariety of $\mathbb{A}^{2n}$ minimal if it's minimal among such.

Theorem 4.19. *Let $d \in \mathcal{D}_n$, such that $\sigma(d)$ is irreducible, and $Z(\sigma(d))$ is coisotropic and minimal. Then the left ideal $\mathcal{D}_n d$ is maximal, so that $\mathcal{D}_n/\mathcal{D}_n d$ is irreducible of dimension $2n-1$ over $\mathcal{D}_n$.*

Proof.

$$0 \to \mathcal{D}_n d \to \mathcal{D}_n \to \mathcal{D}_n/\mathcal{D}_n d \to 0$$

$$0 \to \mathrm{Gr}\, \mathcal{D}_n d \to \mathbb{K}[x_1, \ldots, x_{2n}] \to \mathrm{Gr}\,(\mathcal{D}_n/\mathcal{D}_n d) \to 0$$

$$\mathrm{Ann}\, \mathrm{Gr}\,(\mathcal{D}_n/\mathcal{D}_n d) \simeq \mathbb{K}[x_1, \ldots, x_n]\, \sigma(d)$$

Assume that $\mathcal{D}_n d \subset J$ for some $J \neq \mathcal{D}_n$. Then

$$0 \to J/\mathcal{D}_n d \to \mathcal{D}_n/\mathcal{D}_n d \to \mathcal{D}_n/J \to 0$$

By the minimality of $Z(\sigma(d))$,

$$Z(\sigma(J)) = Z(\sigma(d))$$

Thus $\mathrm{rad}\,\langle\sigma(J)\rangle = \mathrm{rad}\,\langle\sigma(d)\rangle = \langle\sigma(d)\rangle$, hence $J = \langle d\rangle$. □

Theorem 4.20 (Bernstein-Luntz)**.** *The property $\{Z(f)$ is minimal$\}$ holds generically.*

5. Operations on D-modules

We will now define several operations on D-modules, and show that they preserve holonomicity.

1. Fourier transform maps Schwartz functions into Schwartz measures and vice versa. It also maps tempered generalized functions to tempered distributions. It also maps product into convolution and

$$\widehat{x_j f} = (i/2\pi)\partial_j \hat{f}, \quad \widehat{\partial_j f} = 2\pi i x_j \hat{f}.$$

The corresponding operation on $\mathcal{D}_n$-modules is just switching the actions of x_j and ∂_j.

Let us give an application to PDE. Let d be a differential operator on $\mathbb{R}^n$ with constant real coefficients, and h be a smooth function. We are looking for a solution of the equation $df = h$. First of all, it is enough to find a solution for $d\xi = \delta_0$ in distributions, because then the convolution $\xi * h$ will solve the original equation. Now, applying Fourier transform we get the equation $pg = 1$, where p is the polynomial obtained from d by replacing all ∂_j by $2\pi i x_j$, and g is the unknown generalized function. Then it is clear for us that g should be p^{-1}. This is not well-defined a-priori, since p might have zeros. However, $(p^2)^\lambda$, as we have shown, is defined as a meromorphic distribution-valued function in λ. It might have a pole at $\lambda = -1/2$, but then we take the principal part (the lowest non-zero coefficient in the Laurent expansion).

2. One can multiply a distribution by a smooth function. Formally, the result is given by $f\xi(h) := \xi(fh)$. The corresponding operation on $\mathcal{D}_n$-modules is tensor product over $\mathcal{O}_n := \mathcal{O}(A^n) = k[x_1, \dots, x_n]$. Note that a product of a smooth function and a generalized function (= functional on smooth measures) is a generalized function, a product of a function and a distribution is a distribution, and a product of a smooth measure and a distribution is not defined.

Similarly, a product of two left $\mathcal{D}_n$-modules is a left $\mathcal{D}_n$-module, a product of a left $\mathcal{D}_n$-module by a right $\mathcal{D}_n$-module is a right $\mathcal{D}_n$-module, and a product of right $\mathcal{D}_n$-modules is not defined. The $\mathcal{D}_n$-module structure of a product of two (left) $\mathcal{D}_n$-modules is defined via Leibnitz rule:

$$\partial_i(m \otimes n) = \partial_i m \otimes n + m \otimes \partial_i n.$$

One can always turn a left $\mathcal{D}_n$-module to a right one using tensor product with the (right) $\mathcal{D}_n$-module of (algebraic) top differential forms.

3. For a polynomial map of affine spaces $\pi : X \to Y$, we can pullback smooth functions from Y to X. If the map is submersive then we can even pullback generalized functions. Let us define pullback of $\mathcal{D}_n$-modules as well. Let M be an $\mathcal{D}_Y$-module. As an $\mathcal{O}_X$-module we define $\pi^0(M) := \mathcal{O}_X \otimes_{\mathcal{O}_Y} M$. The action of the vector fields $\mathcal{T}_Y$ is defined using the natural morphism $\mathcal{T}_X \to \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{T}_Y$, which on every fiber is defined using $d\pi$. In coordinates:

$$\xi(f \otimes m) = \xi(f) \otimes m + \sum_i f\xi(\pi^*(y_i)) \otimes \partial_i m.$$

By the well-known properties of pullback of $\mathcal{O}_X$-modules we get that $(\tau\pi)^0 = \pi^0\tau^0$, and that pullback is strongly right-exact, i.e. right-exact and commutes with arbitrary direct sums.

Exercise 5.1. Let A,B be rings. Let $F : \mathcal{M}(A) \to \mathcal{M}(B)$ be a strongly right-exact functor. Then $F(A)$ has a natural structure of a $B-A$-bimodule and F is isomorphic to the functor $M \mapsto F(A) \otimes_A M$.

Notation 5.2. $\mathcal{D}_{X\to Y} := \pi^0(\mathcal{D}_Y)$.

Remark 5.3. The intuition here is that $\mathcal{D}_{X\to Y}$ is the $(\mathcal{D}_X, \mathcal{D}_Y)$-bimodule of $\mathcal{O}_X$-valued differential operators on $\mathcal{O}_Y$. For a general commutative algebra A and an A-module M an $\leq n$-th degree differential operator on A with values in M is a $\mathbb{K}$-linear operator $D : A \to M$, such that $[a_1, [a_2, \dots, [a_{n+1}, D]]] = 0$ for all $a_1, \dots, a_{n+1} \in A$. Apparently, in nice cases $\mathcal{D}_{X\to Y}$, defined this way, coincides as an $(\mathcal{O}_X, \mathcal{D}_Y)$-bimodule with $\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{D}_Y$. From here it follows that for any $\mathcal{D}_Y$-module M we have $\mathcal{D}_{X\to Y} \otimes_{\mathcal{D}_Y} M \simeq \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \otimes_{\mathcal{D}_Y} M \simeq \mathcal{O}_X \otimes_{\mathcal{O}_Y} M$.

Lemma 5.4. *For two morphisms $\nu : X \to Y$ and $\mu : Y \to Z$ we have $(\mu\nu)^0 = \nu^0 \circ \mu^0$.*

Proof. It is enough to show that the natural map $\mathcal{D}_{X\to Y} \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to Z} \to \mathcal{D}_{X\to Z}$ is an isomorphism. Since $\mathcal{D}_{X\to Y} \simeq \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{D}_Y$ and $\mathcal{D}_{Y\to Z} \simeq \mathcal{O}_Y \otimes_{\mathcal{O}_Z} \mathcal{D}_Z$, we have

$$\mathcal{D}_{X\to Y} \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to Z} \simeq \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \otimes_{\mathcal{D}_Y} \mathcal{O}_Y \otimes_{\mathcal{O}_Z} \mathcal{D}_Z \simeq \mathcal{O}_X \otimes_{\mathcal{O}_Z} \mathcal{D}_Z \simeq \mathcal{D}_{X\to Z}$$

□

Theorem 5.5 (Bernstein)**.** *The pullback of a holonomic $\mathcal{D}_Y$-module is a holonomic $\mathcal{D}_X$-module.*

We divide the proof into several lemmas.

Lemma 5.6. *Any map $\pi : X \to Y$, where $Y \simeq \mathbb{A}^n$, can be decomposed into a standard embedding, an isomorphism, and a standard projection.*

Proof. Take the maps $X \to X \times Y \to X \times Y \to Y$, $x \mapsto (x, 0)$, $(x, y) \mapsto (x, y + \pi(x))$, $(x, y) \mapsto y$. □

Lemma 5.7. *Let T, Y be affine spaces. The pullback under the standard projection $p : T \times Y \to Y$ of a holonomic module is holonomic.*

Proof. In this case the pullback is the exterior product $\mathcal{O}_T \otimes_k M$. It is easy to see that exterior product of holonomic modules is holonomic. □

Lemma 5.8. *The pullback under an isomorphism $i : X \to Y$ of a holonomic $\mathcal{D}_Y$-module is a holonomic $\mathcal{D}_X$-module.*

Proof. In this case we can consider the pullback as the same space, just a different action. If F^iM is a good filtration for the original action and $r := \deg \pi$, then $\Phi^i M := F^{ri}M$is a filtration for the new action, and it satisfies $dim\Phi^i M \leq (cr^d)i^d$. □

Lemma 5.9. *The pullback under the standard embedding $i : X \to X \times \mathbb{A}^1$ of a holonomic $\mathcal{D}_{X\times\mathbb{A}^1}$-module is a holonomic $\mathcal{D}_X$-module.*

This lemma is the difficult one. Indeed, in this case the pullback of a finitely-generated module might be not finitely-generated.
For example, for $X = pt$ we get $i^0(\mathcal{D}_{X\times\mathbb{A}^1}) = D_1$.

To prove the lemma, we will need another important lemma, that we in fact partially proved in the first lecture.

Lemma 5.10 (Kashiwara)**.** *Let N be a $\mathcal{D}_{X\times\mathbb{A}^1}$-module. Denote by t the coordinate of $\mathbb{A}^1$. Assume that t acts locally nilpotently on N and let $R := \operatorname{Ker} t$, $R_i := \partial_t^i R$. Then $N = \bigoplus_i R_i$ and $t\partial_t$ acts on R_i by the scalar $-(i+1)$.*

Proof. 1. We note that $(\partial_t t + i)\,\partial_t^i(\ker t) = 0$.
Indeed, $[t, \partial_t^i] = -i\partial_t^{i-1}$, so $(\partial_t t + i)\,\partial_t^i = \partial_t^{i+1} t$.

2. $t\partial_t^i(\ker t) = -i\partial_t^{i-1}(\ker t)$. Thus $\partial_t^i \ker t \subset \ker t^{i+1}$.

3. $(\partial_t t + i)\ker t^{i+1} \subset \ker t^i$. Indeed, for $m \in \ker t^{i+1}$,

$$t\,(\partial_t t + i)\,m = (i-1)\,tm + \partial_t t^2 m = (\partial_t t + i - 1)\,tm \subset \ker t^{i-1}$$

by induction.

4. $\partial_t^i \ker t$ are the different eigenspaces of $\partial_t t$, so their sum is direct: $\bigoplus_i \partial_t^i \ker t$.

Now we show that

$$\ker t^i = \bigoplus_{j=0}^{i-1} \partial_t^j \ker t$$

Take $m \in \ker t^{i+1}$. Then $(\partial_t t + i)\, m \in \ker t^i$, so by induction

$$(\partial_t t + i)\, m \in \bigoplus_{j=0}^{i-1} \partial_t^j \ker t.$$

Again, by induction,

$$tm \in \bigoplus_{j=0}^{i-1} \partial_t^j \ker t,$$

so $\partial_t tm \in \bigoplus_{j=1}^{i} \partial_t^j \ker t$. Thus $m \in \bigoplus_{j=0}^{i} \partial_t^j \ker t$. □

Proof of Lemma 5.9. Let N be a holonomic $\mathcal{D}_{X\times\mathbb{A}^1}$-module. Denote by t the coordinate of $\mathbb{A}^1$. Then

$$M := i^0(N) = N/tN.$$

Denote by N^0 the submodule consisting of elements annihilated by powers of t. Then, by Kashiwara's lemma, we have $tN_0 = N_0$. Thus $i^0(N) = i^0(N')$, where $N' = N/N_0$. Now, N' is also holonomic and t has no kernel on N'. Choose a good filtration F^iN' and define the corresponding good filtration F^iM by projection. Then

$$\dim F^iM \le \dim F^iN' - \dim tF^{i-1}N' = \dim F^iN' - \dim F^{i-1}N' \le ci^{\dim X}.$$

□

Theorem 5.5 follows now from Lemmas 5.6,5.7,5.8, and 5.9.

Corollary 5.11. *If $M, N \in \mathrm{Hol}\,(\mathcal{D}_X)$ then $M \otimes_{\mathcal{O}_X} N \in \mathrm{Hol}\,(\mathcal{D}_X)$.*

Proof. $M \otimes_X N = \Delta^0\,(M \otimes_{\mathbb{K}} N)$, $\Delta : X \to X \times X$ is the diagonal. □

4. For a polynomial map of affine spaces $\pi : X \to Y$, we can pushforward smooth compactly supported measures from Y to X, by integration by fibers. Note that we indeed push measures and not functions. This hints that the pushforward π_0 should be defined for right $\mathcal{D}_X$-modules.

Definition 5.12. For $M \in \mathcal{M}^r(\mathcal{D}_X)$ define $\pi_0(M) := M \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y} \in \mathcal{M}^r(\mathcal{D}_Y)$.

This operation also preserves holonomicity. This can again be shown by decomposing the map into three parts. The difficult case now will be the standard projection. However, we will prove this differently using a trick.

Exercise 5.13. Let $M \in \mathcal{M}^r(\mathcal{D}_V)$ and let $\mathcal{F}(M) \in \mathcal{M}^l(\mathcal{D}_{V^*})$ denote the module obtained from M by swapping the actions of x_i and ∂_i. Let $T : V \to W$ be a linear map, and let $T^* : W^* \to V^*$ denote the dual map. Then $\mathcal{F}(T_0M) = (T^*)^0(\mathcal{F}(M))$.

Corollary 5.14. *Pushforward of a holonomic module is holonomic.*

Proof. For isomorphisms it is easy. The standard embeddings and projections are linear maps, and thus for them it follows from Exercise 5.13 and Theorem 5.5. □

Exercise 5.15. For an isomorphism ν, $\nu_0 = (\nu^{-1})^0$.

Remark 5.16. One can also define different versions of pullback and pushforward, by $Hom_{\mathcal{D}_X}(\mathcal{D}_{X\to Y}, M)$. These functors will be right adjoint to the functors we defined.

Similarly to Lemma 5.4, we have

Lemma 5.17. *For two morphisms $\nu : X \to Y$ and $\mu : Y \to Z$ we have $(\nu\mu)_0 = \nu_0 \circ \mu_0$.*

Let us now examine how does pushforward look like. For $p : \mathbb{A}^1 \to pt$ we have $p_0(M) = M/M\partial_t$. For $i : pt \hookrightarrow \mathbb{A}^1$ we have $i_0(k) := \bigoplus k\delta^i$, with $\delta^i\partial_t = \delta^{i+1}$and $\delta^i t = i\delta^{i-1}$.

Example 5.18. $p : \mathbb{A}^1 \to \{\mathrm{pt}\}$. $p_0(M) = M/M\partial_t$.
For $i : \{\mathrm{pt}\} \to \mathbb{A}^1$: $i^0(M) := \bigoplus \mathbb{K}\delta^{(i)}$ is the $\mathcal{D}$-module of distributions supported at $\{\mathrm{pt}\}$. $\delta^{(i)}\partial_t := \partial^{(i+1)}$, $\delta^{(i)}t := i\delta^{(i-1)}$.

Lemma 5.19. *Let $\xi \in S^*(\mathbb{R}^n)$ be a tempered distribution, p be a positive polynomial, and $p \to \infty$ at ∞. Then $\lambda \mapsto \langle \xi, p^\lambda \rangle$ converges for $\Re\lambda < -r$ for some r.*

Lemma 5.20. *Let ξ be holonomic. Then there exist rational functions $q_1, \ldots, q_\ell \in \mathbb{C}(\lambda)$, such that*

$$\langle \xi, p^\lambda \rangle = \sum_i q_i \langle \xi, p^{\lambda - i} \rangle$$

Proof. Take the $\mathcal{D}$-module M generated by the distribution $p^{\lambda-k}\xi$ over the field $\mathbb{C}(\lambda)$. It is holonomic. Thus its pushfoward to the point is holonomic. On the other hand, the pushforward to the point is $M/\partial M$ $(\mathcal{D}_{X\to\mathrm{pt}} = \mathcal{O}(X))$. Being holonomic over a point means that it's a finite-dimensional vector space over $\mathbb{C}(\lambda)$. Thus $p^{\lambda-k}\xi$ are $\mathbb{C}(\lambda)$-linearly dependent modulo ∂M. The integral on ∂M vanishes, thus $\int \xi p^\lambda$ satisfies this linear dependence. ∎

6. Homological properties

Let $\mathcal{C}$ be an abelian category.

Definition 6.1. We say that $\mathcal{C}$ has homological dimension $\le d$ if for any $M \in \mathcal{C}$ and any projective resolution

$$P_{d-1} \to \cdots \to P_0 \to M \to 0,$$

the kernel $\ker(P_{d-1} \to P_{d-2})$ is projective.

Theorem 6.2. *The following are equivalent for $\mathcal{C}$:*
1. *Any object has a projective resolution of length $\le d$.*
2. Ext^{d+i} *vanishes for all $i \ge 1$.*
3. *The derived functor $L_{d+i}F$ vanishes for any right exact functor F and all $i \ge 1$..*
4. $\mathrm{hd}\,\mathcal{C} \le d$.

Definition 6.3. Let V be a vector space, and let $a_1, \ldots, a_n : V \to V$ be commuting operators. The Koszul complex of $C(V, a_1, \ldots, a_n)$ is the complex (numbered by $n, \ldots, 0$)

$$0 \to \Lambda^n\mathbb{K}^n \otimes V \to \Lambda^{n-1}\mathbb{K}^n \otimes V \to \cdots \to \Lambda^0\mathbb{K}^n \otimes V \to 0$$

with differential $\sum_i \frac{\partial}{\partial\xi_i} \otimes a_i$, where $\frac{\partial}{\partial\xi_i}$ is the interior product with the basis vector ξ_i.

Definition 6.4. A sequence (a_i) is regular if a_i has no kernel on $V/\left(a_1V+\cdots+a_{i-1}V\right)$, for all i.

Theorem 6.5 (Proof-Exercise)**.** *If the sequence (a_i) is regular then the Koszul complex is acyclic outside* 0*, and*

$$H_0\left(C\right)\simeq V/\left(a_1V+\cdots+a_nV\right).$$

Let $A:=\mathbb{K}\left[x_1,\ldots,x_n\right]$.

Theorem 6.6 (Hilbert's syzygy)**.** *The homological dimension of $A=\mathbb{K}\left[x_1,\ldots,x_n\right]$ is n.*

Proof. The Koszul complex of $x_1,\ldots,x_n$ acting on A is a free resolution of the module $A/\left(x_1,\ldots,x_n\right)A$. For an arbitrary module M let x_i act on $A\otimes_{\mathbb{K}}M$ by

$$x_i\left(a\otimes m\right):=x_ia\otimes m+a\otimes x_im.$$

This defines an A-module structure on $A\otimes_{\mathbb{K}}M$. This module is free (exercise). Thus the complex

$$C\left(A,x_1,\ldots,x_n\right)\otimes_{\mathbb{K}}M$$

is a free resolution of M. □

Lemma 6.7 (Graded Nakayama's lemma)**.** *Let M be a finitely generated graded A-module with $M=\left(x_1,\ldots,x_n\right)M$. Then $M=0$.*

Proof. Since M is f.g. and $M=\left(x_1,\ldots,x_n\right)M$, the Nakayama's lemma implies that $0\notin\operatorname{supp}M$. Since M is graded, $\operatorname{supp}M$ is conical and thus empty. □

Corollary 6.8. *Any graded projective finitely generated module P over $\mathbb{K}\left[x_1,\ldots,x_n\right]$ is free.*

Proof. Let $\mathfrak{m}:=\left(x_1,\ldots,x_n\right)$. Choose homogeneous elements $p_i\in P$ such that their projections to $P/\mathfrak{m}P$ form a basis. By the graded Nakayama's lemma, p_i generate P. Thus we have a s.e.s. of graded modules $0\to K\to A^m\to P\to 0$. Here A^m has its grading shifted according to the degrees of p_i. Since P is projective, this sequence splits. So $A^m\simeq K\oplus P$. Thus $K/\mathfrak{m}K=0$, so $K=0$. □

Corollary 6.9. *Any graded finitely generated A-module has a free graded resolution of length $\leq n$.*

Definition 6.10. For a Noetherian ring R we denote by $\mathcal{M}^f(R)$ the category of finitely-generated left R-modules, and by $\operatorname{hd}(R)$ the homological dimension of this category.

Exercise 6.11 (*)**.** $\operatorname{hd}(\mathcal{M}(R))=\operatorname{hd}(R)$.

Exercise 6.12. Let R be a ring and $M\in\mathcal{M}^f(R)$ with a good filtration. Then

(i) for some l there exists a good filtration on R^l and a strict epimorphism $R^l\twoheadrightarrow M$.
(ii) If $\operatorname{Gr}M$ is free then M is free.

From Corollary 6.9 we obtain

Corollary 6.13. $\operatorname{hd}\mathcal{D}_n\leq 2n$.

Proof. Let $M \in \mathcal{M}^f(\mathcal{D}_n)$. Choose a good filtration on M. By Exercise 6.12(i) there exists a free $\mathcal{D}_n$-module F_1 with good filtration and a strict epimorphism $\varphi_1 : F_1 \twoheadrightarrow M$. Let L_1 be the kernel of φ_1 with induced filtration and choose a free F_2 again using Exercise 6.12(i). Continuing in this way we obtain an exact sequence of finitely-generated filtered modules with strict maps:

$$0 \to L_{2n-1} \to F_{2n-1} \to \cdots \to F_1 \to M \to 0,$$

with F_i free. By Exercise 1.20, the associated graded sequence

$$0 \to \operatorname{Gr} L_{2n-1} \to \operatorname{Gr} F_{2n-1} \to \cdots \to \operatorname{Gr} M \to 0$$

is also exact. By Hilbert's syzygy theorem, $\operatorname{Gr} L_{2n-1}$ is projective, and thus free. By Exercise 6.12(ii), L_{2n-1} is free and the above sequence is a free resolution of M of length $2n$. $\square$

Now we want to show that $\operatorname{hd} \mathcal{D}_n = n$.

Corollary 6.14. *Let R be a Noetherian ring with $\operatorname{hd} R < \infty$, and M be a finitely generated R-module. Then $\operatorname{hd} M \leq d$ iff $\operatorname{Ext}^i(M, R) = 0 \,\forall i > d$.*

Proof. Let $M \in \mathcal{M}^f(R)$ with $\operatorname{Ext}^i(M, R) = 0 \,\forall i > d$.

We have to show that $\operatorname{Ext}^i(M, \cdot) = 0 \,\forall i > d$.

Take any finitely generated X, and consider $0 \to L \to R^\ell \to X \to 0$. Thus:

$$\operatorname{Ext}^i\left(M, R^\ell\right) \to \operatorname{Ext}^i(M, X) \to \operatorname{Ext}^{i+1}(M, L) \to \operatorname{Ext}^{i+1}\left(M, R^\ell\right)$$

Thus for $i > d$ we have $\operatorname{Ext}^i(M, X) \simeq \operatorname{Ext}^{i+1}(M, L)$.

By induction on i descending from $\operatorname{hd} M$, $\operatorname{Ext}^{>d}(M, \cdot) = 0$. $\square$

Let $A := \mathbb{K}[x_1, \ldots, x_n]$, M be a f.g. A-module. Denote $E^i(M) := \operatorname{Ext}^i(M, A)$.

Theorem 6.15 (Serre ??)**.** *Let $d := d(M)$. Then $E^i(M) = 0, \forall i < n - d$.*

Proof. Induction on n. Take $B := \mathbb{K}[x_1, \ldots, x_{n-1}]$. If M is finitely generated over B, take

$$N := M[t] \simeq A \otimes_B M$$

Then $E^i(N) \simeq \operatorname{Ext}^i(M, B)[t]$. Thus $E^i(N) = 0$ for $i < n - 1 - d$, and $E^{n-1-d}(N)$ is free over $\mathbb{K}[t]$. Now we have a s.e.s.

$$0 \to N \overset{t-x_n}{\to} N \to M \to 0$$

Thus

$$E^{n-2-d}(N) \to E^{n-1-d}(M) \overset{0}{\to} E^{n-1-d}(N) \overset{t-x_n}{\to} E^{n-1-d}(N)$$

The rightmost map has trivial kernel, so the arrow in the middle is 0. Now $E^{n-2-d}(N) = 0$ implies $E^{n-1-d}(M) = 0$.

Now we treat the general case when M is not necessarily finitely generated over B. If $d(M) = n$ then there is nothing to prove, otherwise by Noether's normalization lemma there exists a linear coordinate change $y_i = Tx_i$ such that $A/\operatorname{Rad}(\operatorname{Ann}(M))$ is finite over $\mathbb{K}[y_1, \ldots, y_{n-1}]$. Then M is finitely generated over $\mathbb{K}[y_1, \ldots, y_{n-1}]$, and we reduce to the previous case. $\square$

Lemma 6.16. *Assume that M is graded. Then*
$$d(M) \leq d(\operatorname{coker}(x_n|_M)) + 1.$$

Proof.
$$M^i \overset{x_n}{\to} M^{i+1} \to (\operatorname{coker}(x_n|_M))^{i+1} \to 0$$
Thus $\Delta d_M(i) \leq d_{\operatorname{coker} x_n}(i+1)$. ☐

Corollary 6.17. *Assume that* $\ker(x_n \restriction_M) = 0$. *Then*
$$d(M) \leq d(\operatorname{coker}(x_n|_M)) + 1.$$

Proof. $0 \to M \overset{x_n}{\to} M \to \operatorname{coker} x_n \to 0$. Introduce a filtration on M, pass to the associated graded module.
$$0 \to \operatorname{Gr} M \to \operatorname{Gr} M \to \operatorname{Gr} \operatorname{coker} x_n \to 0$$
Note that $\operatorname{Gr}\operatorname{coker} x_n = \operatorname{coker}(x_n \restriction_{\operatorname{Gr} M})$, and use the lemma on graded modules. ☐

Corollary 6.18. $d(M) \leq \max(d(\ker(x_n|_M)), d(\operatorname{coker}(x_n|_M)) + 1)$.

Proof. We can assume that $d(\ker(x_n \restriction_M)) < d(M)$. Then $d(\bigcup_i \ker(x_n^i|_M)) < d(M)$. Indeed, $\bigcup_i \ker(x_n^i \restriction_M)$ stabilizes at a finite union, and
$$\ker\left(x_n^i|_M\right) / \ker\left(x_n^{i-1}|_M\right) \simeq \ker\left(x_n|_{M/\ker\left(x_n^{i-1}\restriction_M\right)}\right),$$
whose d is $< d(M)$. Thus $d(M) = d(N)$, where
$$N := M/\bigcup_i \ker x_n^i$$
Now $\ker(x_n \restriction N) = 0$, so we reduce to the previous corollary. ☐

Lemma 6.19. $\operatorname{supp} E^i(M) \subset \operatorname{supp} M$

Proof. $\operatorname{Ann} M \subset \operatorname{Ann} E^i M$. ☐

Theorem 6.20 (Ross ??)**.** *For any $M \in \mathcal{M}^f(A)$, $d(E^i M) \leq n - i$.*

Proof. We prove by induction on n. Consider first the case when M is finitely generated over $B = \mathbb{K}[x_1, \ldots, x_{n-1}]$. $N := M[t] \simeq A \otimes_B M$, $E^i(N) = \operatorname{Ext}^i(M, B)[t]$ thus by the induction hypothesis
$$d\left(E^i(N)\right) \leq n - 1 - i + 1 = n - i$$
Now,
$$0 \to N \overset{t-x_n}{\to} N \to M \to 0$$
Thus
$$\cdots \to E^{i-1}(M) \overset{0}{\to} E^{i-1}(N) \to E^{i-1}(N) \to E^i(M) \to E^i(N) \to E^i(N) \to \ldots$$
The map $E^{i-1}(M) \to E^{i-1}(N)$ is 0 because $t - x_n$ has zero kernel.

For any $v \in E^{i-1}(M)$, $(t - x_n)v = 0$, but $t - x_n$ has no kernel in $E^{i-1}(N)$ because $E^i(N) = \operatorname{Ext}^{i-1}(N, B)[t]$ and $t - x_n$ shifts the degree by 1.

Now introduce a filtration on $E^{i-1}(N)$ that is a grading in t. Then
$$0 \to F^j\left(E^{i-1}N\right) \to F^{j+1}\left(E^{i-1}N\right) \to F^{j+1}\left(E^i M\right) \to 0$$

Thus $d\left(E^{i}M\right)=d\left(E^{i-1}N\right)-1\leq n-(i-1)-1=n-i$.

The next case is that $x_n : M \to M$ is injective. Then

$$0\to M\xrightarrow{x_n}M\to L\to 0$$

Thus

$$E^{i}L\to E^{i}M\xrightarrow{x_n}E^{i}M\to E^{i+1}L$$

By the last corollary, $d\left(E^{i}M\right)\leq\max\left(d\left(E^{i}L\right),d\left(E^{i+1}L\right)+1\right)\leq n-i$.

Finally, in the general case

$$0\to K\to M\to L\to 0$$

where x_n is nilpotent on K and $x_n\restriction_L$ is bijective Thus

$$\dots\to E^{i}L\to E^{i}M\to E^{i}K\to\dots$$

Thus $d\left(E^{i}M\right)\leq\max\left(d\left(E^{i}L\right),d\left(E^{i}K\right)\right)\leq n-i$ by the previous cases. $\square$

Corollary 6.21. *Let* $M\in\mathcal{M}^{f}\left(\mathcal{D}_n\right)$. *Then*

(1) $\operatorname{Ext}^{i}\left(M,\mathcal{D}_n\right)=0$, $\forall i<2n-d\left(M\right)$

(2) $2n-d\left(\operatorname{Ext}^{i}\left(M,\mathcal{D}_n\right)\right)\geq i$

Proof. As we proved before, M has a resolution of length $2n$ consisting of free finitely generated filtered modules and strict maps:

$$0\to F_{2n}\to\dots\to F_0\to 0$$

Taking Hom into $\mathcal{D}_n$ we get

$$0\to F_0^{*}\to\dots\to F_{2n}^{*}\to 0$$

Passing to associated graded we have

$$0\to\operatorname{Gr}F_0^{*}\to\dots\to\operatorname{Gr}F_{2n}^{*}\to 0$$

The cohomologies of the latter sequence are isomorphic both to $\operatorname{Ext}^{i}(\operatorname{Gr}M,A)$ and to $\operatorname{Gr}(\operatorname{Ext}^{i}(M,\mathcal{D}_n))$. The statements now follow from Theorems 6.15 and 6.20. $\square$

Corollary 6.22.

(i) $\operatorname{hd}\mathcal{M}^{f}\left(\mathcal{D}_n\right)\leq n$.

(ii) For any $M\in\mathcal{M}^{f}\left(\mathcal{D}_n\right)$, $\operatorname{Ext}^{n}\left(M,\mathcal{D}_n\right)$ *is holonomic.*

(iii) For a holonomic module $\operatorname{Ext}^{<n}\left(M,\mathcal{D}_n\right)=0$.

Proof. $d\left(E^{n+i}M\right)\leq n-i$, so by Bernstein's inequality, $E^{n+i}M=0$ for $i>0$. For $i=0$ we get $d\left(E^{n}M\right)\leq n$. For holonomic M we have $n-i<2n-d(M)$ and thus $\operatorname{Ext}^{n-i}(M,\mathcal{D}_n)=0$ for any $i>0$. $\square$

Definition 6.23. Define $D:\operatorname{Hol}^{\ell}\left(\mathcal{D}_n\right)\to\operatorname{Hol}^{r}\left(\mathcal{D}_n\right)$ by $\operatorname{Ext}^{n}\left(\cdot,\mathcal{D}_n\right)$.

Theorem 6.24. *D is an equivalence of categories, and* $D\circ D\simeq\mathrm{id}$.

Proof. To prove that $D\circ D\simeq\mathrm{id}$, take a free resolution of M:

$$0\to F_n\to\dots\to F_0\to 0$$

and dualize it by $\operatorname{Hom}\left(\cdot,\mathcal{D}_n\right)$. Since M doesn't have smaller Ext's, this will be a free resolution of DM. Since DM does not have smaller Ext's either, the "double dual" of the resolution of M will be a resolution of $D(D(M))$. Since any free $\mathcal{D}_n$-module F is canonically isomorphic to $\mathrm{Hom}(\mathrm{Hom}(F, \mathcal{D}_n), \mathcal{D}_n)$, we get that $D(D(M)) \cong M$.

Finally, $D \circ D \simeq \mathrm{id}$ implies that D is an equivalence of categories. □

Note that we have only used that $\mathrm{Ext}^i(M, \mathcal{D}_n) = 0$ for any $i < n$. Thus, the same proof shows the following corollary.

Corollary 6.25. *M is holonomic if and only if $\mathrm{Ext}^i(M, \mathcal{D}_n) = 0$ for any $i < n$.*

Remark 6.26. Everywhere in this section we could have used the geometric filtration on $\mathcal{D}_n$ instead of the Bernstein filtration. This gives another proof that modules holonomic with respect to the Bernstein filtration are holonomic with respect to the geometric filtration, and vice versa.

Theorem 6.27. *For any $M \in \mathcal{M}^f(\mathcal{D}_n)$ there is a canonical embedding*

$$0 \to D\left(\mathrm{Ext}^n(M, \mathcal{D}_n)\right) \to M$$

Moreover, its image is the maximal holonomic submodule of M.

Proof. $H = \mathrm{Ext}^n(M, \mathcal{D}_n)$. Let $0 \to P_n \to \cdots \to P_0 \to 0$ be a free resolution of M. Dualize it:

$$0 \to P_0^* \to \cdots \to P_n^* \to 0$$

Now consider a free resolution of H:

$$0 \to Q_n \to \cdots \to Q_0 \to 0$$

H is the last cohomology of P^*, so it is a factor of P_n^*. Now step by step we lift this to a map of complexes $P^* \to Q$. Dualizing, we get maps $Q^* \to P$. By Corollary 6.22, H is holonomic and thus $\mathrm{Ext}^{<n}(H, \mathcal{D}_n)$ vanish. Thus Q^* is a resolution of DH. Thus we get a map $DH \to M$ whose image is a holonomic submodule of M.

The map $DH \to M$ is injective because the right-exact functor $\mathrm{Ext}^n(\cdot, \mathcal{D}_n)$ maps it to the identity map. Finally, for any holonomic submodule $L \subset M$ we have an onto map $H \twoheadrightarrow DL$ and thus an embedding $L \subset DH$. □

We remark that we did not have to use free resolutions in the proofs. Any projective resolution would work, because projective modules are direct summands of free ones.

Exercise 6.28. Let $L := k[x, x^{-1}]$, $M := k[x]$ and $C := L/M$. Note that they are all holonomic and consider the exact sequence $0 \to M \to L \to C \to 0$.

Compute the dual D-modules, and describe the dual exact sequence

$$0 \to D(C) \to D(L) \to D(M) \to 0$$

in terms of distributions.

7. D-modules on smooth affine varieties

First of all, let us give a coordinate-free definition of the algebra of differential operators $\mathcal{D}(V)$ for any vector space V. This is the $\mathbb{K}$-algebra with 1 generated over $\mathbb{K}$ by linear functionals on V and by symbols $\{\partial_v \,|v \in V\}$, with the commutation relations $[\partial_v, \xi] = \xi(v)$.

Let us now define the notion of smoothness for affine algebraic varieties.

Let $X := \operatorname{Spec} A$ be an affine algebraic variety, and for $x \in X$ let $\mathbb{K}_x := A/\mathfrak{m}_x$. $T_x^* X := \mathfrak{m}_x/\mathfrak{m}_x^2$. We start with several well-known definitions and theorems from algebraic geometry. Denote by $\mathcal{O}$ the sheaf of regular functions on X. In particular, $\mathcal{O}(X) = A$.

Theorem 7.1. *The following are equivalent:*

(1) $\mathbb{K}_x$ *has finite homological dimension as an A-module.*
(2) $\operatorname{Gr}_{\mathfrak{m}_x} A := \bigoplus (\mathfrak{m}_x^i/\mathfrak{m}_x^{i+1})$ *is a polynomial algebra.*
(3) $\dim T_x^* X = \dim_x X$.
(4) *Locally around* x *there is a quasi-coordinate system.*

Definition 7.2. We say that X is smooth at x if these conditions hold.

Definition 7.3. A quasi-coordinate system of an affine variety U at $x \in U$ is:

(1) A collection of functions $x_1, \dots, x_n \in \mathcal{O}(U)$;
(2) a collection of vector fields $\partial_1, \dots, \partial_n \in \operatorname{Der} \mathcal{O}(U)$,

such that
(a) $\partial_i x_j = \delta_{ij}$;
(b) dx_i span $T_u^* U$ for all $u \in U$.

Theorem 7.4. *The set of smooth points is open and dense.*

Definition 7.5. Let $\tau_X := \operatorname{Der} \mathcal{O}(X)$ denote the Lie algebra of derivations of $\mathcal{O}(X)$, i.e. linear endomorphisms of $\mathcal{O}(X)$ satisfying the Leibnitz rule: $\partial(fg) = \partial(f)g + f\partial(g)$. Elements of $\operatorname{Der} \mathcal{O}(X)$ are called (algebraic) vector fields on X.

Definition 7.6. $\mathcal{D}^{\leq -1}(X) := 0$,

$$\mathcal{D}^{\leq k}(X) := \left\{ d \in \operatorname{Hom}_{\mathbb{K}}(\mathcal{O}(X), \mathcal{O}(X)) \;\middle|\; \forall f \in \mathcal{O}(X) : [f, d] \in \mathcal{D}^{\leq k-1}(X) \right\}.$$

Similarly, for $\mathcal{O}(X)$-modules M, N define $\mathcal{D}^{\leq k}(M, N)$.

Example 7.7. $\mathcal{D}^{\leq 0}(X) = \mathcal{O}(X)$, $\mathcal{D}^{\leq 1}(X) = \mathcal{O}(X) \oplus \operatorname{Der} \mathcal{O}(X)$.

The algebra of algebraic differential operators is defined by $\mathcal{D}(X) := \bigcup_i \mathcal{D}^i(X)$. We will show that if X is smooth then $\mathcal{D}(X)$ is Noetherian and generated by $\mathcal{O}(X)$ and $\operatorname{Der} \mathcal{O}(X)$.

Exercise 7.8 (*)**.**

(1) If $X = \{\sum_i x_i^2 = 0\}$ then $\mathcal{D}(X)$ is Noetherian but not generated by $\mathcal{D}^{\leq 1}(X)$.
(2) If $X = \{\sum_i x_i^3 = 0\}$ then $\mathcal{D}(X)$ is not Noetherian.

From now on we assume that X is smooth.

Theorem 7.9. *Let* $M, N \in \mathcal{M}(\mathcal{O}(X))$, *let* $d \in \mathcal{D}^{\leq k}(M, N)$ *and* $f \in \mathcal{O}(X)$. *Then it uniquely defines* $d' \in \mathcal{D}^{\leq k}(M_f, N_f)$.

Proof. Define $d(f^{-i} m)$ by induction on i and k:

$$d'\left(f^{-i} m\right) := f^{-1} d'\left(f^{-i+1} m\right) - f^{-1}[d', f]\left(f^{-i} m\right)$$

□

Corollary 7.10. *If* M *is finitely generated then* $(\mathcal{D}(M, N))_f \simeq \mathcal{D}_{\mathcal{O}_f}(M_f, N_f)$.

Proof. To construct the map in the only nontrivial direction $\mathcal{D}_{\mathcal{O}_f}(M_f, N_f) \to (\mathcal{D}(M,N))_f$, take the common denominator of the generators of M. $\square$

Definition 7.11. Define the sheaf $\mathcal{D}_X$ of differential operators on X by

$$\mathcal{D}_X(X_f) := \mathcal{D}(X_f).$$

By Corollary 7.10, $\mathcal{D}_X$ is a quasi-coherent sheaf.

Remark 7.12. In general, a good calculus of fractions is guaranteed by the Ore condition. For a ring A and a multiplicative set S, the Ore condition is that for any $a \in A, s \in S$ there are $a' \in A, s' \in S$, such that $as' = sa'$ (i.e. $s^{-1}a = a's'^{-1}$). For $S = \{f^n\}, f \in \mathcal{O}_X$, $A = \mathcal{D}_X$, it is satisfied.

Recall that τ_X denotes the tangent sheaf of X. Note that the existence of a quasi-coordinate system implies that τ_X is coherent and locally free. Let $\operatorname{Sym}\tau_X(X)$ denote the symmetric algebra $\bigoplus_i \operatorname{Sym}^i(\tau_X)$, where $\operatorname{Sym}^i(\tau_X)$ denotes the module of symmetric tensors in $\tau_X \otimes_{\mathcal{O}(X)} \tau_X \cdots \otimes_{\mathcal{O}(X)} \tau_X$. Let $T^*X := \operatorname{Spec}\operatorname{Sym}\tau_X(X)$. It is called the cotangent bundle of X.

Theorem 7.13. $\Sigma := \operatorname{Gr}\mathcal{D}(X) \simeq \mathcal{O}(T^*X) := \operatorname{Sym}\tau_X(X)$.

Proof. Define $\Sigma^\ell := \operatorname{Sym}^\ell \tau_X(X)$. For $d \in \mathcal{D}^{\leq \ell}$ consider its symbol

$$(\sigma d)(f_1, \ldots, f_\ell) := [[d, f_1], \ldots, f_\ell].$$

This is a $\mathcal{O}(X)$-valued n-linear form on $\mathcal{O}(X)$, that is a derivation in each variable f_i. There is a canonical map from $\operatorname{Sym}^\ell(\tau_X(X))$ to the space of such forms. Using a quasi-coordinate system, one can show that this map is an isomorphism. Thus, we view σd as an element of $\operatorname{Sym}^\ell(\tau_X(X))$.

Clearly, $d \mapsto \sigma d$ is an embedding. To show that it is onto, just take the product of vector fields to produce a given symbol. $\square$

Corollary 7.14. *$\mathcal{D}(X)$ is Noetherian and $hd(\mathcal{D}(X)) \leq 2\dim X$.*

Exercise 7.15. The structure of a left $\mathcal{D}(X)$-module on an $\mathcal{O}(X)$-module M is the same as the structure of a τ_X-module on M satisfying

$$(f\xi)m = f(\xi m) \text{ and } \xi(fm) - f(\xi m) = \xi(f)m.$$

The structure of a right $\mathcal{D}(X)$-module on an $\mathcal{O}(X)$-module M is the same as the structure of a module over the opposite of the Lie algebra τ_X satisfying

$$(f\xi)m = f(\xi m) \text{ and } \xi(fm) - f(\xi m) = -\xi(f)m.$$

By module over the opposite Lie algebra we mean the identity

$$\xi_1(\xi_2 m) - \xi_2(\xi_1 m) = -[\xi_1, \xi_2]m.$$

Exercise 7.16. The module of top differential forms Ω_X^{top} with the action $\xi\alpha := -Lie_\xi\alpha$ (Lie derivative) is a right $\mathcal{D}(X)$-module. Moreover, $M \mapsto M \otimes_{\mathcal{O}_X} \Omega_X^{top}$ defines an equivalence of categories $\mathcal{M}(\mathcal{D}(X)) \simeq \mathcal{M}^r(\mathcal{D}(X))$.

The push and pull functors are defined for affine varieties in the same way as for affine spaces. Namely, for $\pi : X \to Y$ and $N \in \mathcal{M}(\mathcal{D}(Y))$ define

$$\pi^0(N) := \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} N,$$

with the action of $\tau(X)$ given by the morphism $\tau(X) \to \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \tau(Y)$.
As before, π^0 is strongly right-exact and thus

$$\pi^0(N) = \mathcal{D}_{X\to Y} \otimes_{\mathcal{D}(Y)} N, \text{ where } \mathcal{D}_{X\to Y} = \pi^0(\mathcal{D}(Y)) = \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathcal{D}(Y).$$

For $M \in \mathcal{M}(\mathcal{D}(X))$ we define

$$\pi_0(M) := M \otimes_{\mathcal{D}(X)} \mathcal{D}_{X\to Y}.$$

8. $\mathcal{D}$-modules on general separated smooth varieties

Fact 8.1. *For a variety TFAE:*

(i) For any open affine U, V the intersection $U \cap V$ is affine, and $\mathcal{O}(U) \otimes_{\mathbb{K}} \mathcal{O}(V) \to \mathcal{O}(U \cap V)$ is onto.
(ii) There is an open affine covering (U_i), s.t. the previous property holds for each U_i, U_j
(iii) $\Delta X \subset X \times X$ is closed.

Varieties that satisfy these properties are called separated.

Definition 8.2. Let X be a smooth separated variety. Define the quasi-coherent sheaf of $\mathcal{O}_X$-algebras $\mathcal{D}_X$ by the property $\mathcal{D}_X(U) = \mathcal{D}(U)$ for every open affine $U \subset X$.

A $\mathcal{D}_X$-module is a sheaf of modules over the sheaf of algebras $\mathcal{D}_X$ that is quasi-coherent as a sheaf of $\mathcal{O}_X$-modules. That is, it's a quasi-coherent sheaf $\mathcal{F}$, such that $\mathcal{F}(U)$ have compatible structures of $\mathcal{D}_X(U)$-modules. We will denote the category of $\mathcal{D}_X$-modules by $\mathcal{M}(\mathcal{D}_X)$ and the category of quasi-coherent sheaves by $\mathcal{M}(\mathcal{O}_X)$.

Serre's theorem implies that for an affine X, $\mathcal{M}(\mathcal{D}_X) \simeq \mathcal{M}(\mathcal{D}(X))$.

Definition 8.3. A morphism of algebraic varieties $\pi : X \to Y$ is called affine if $\pi^{-1}(U)$ is affine for any open affine $U \subset Y$.

Example 8.4. Closed embeddings are affine. The embedding of an open affine subset into a separated variety is also affine.

Definition 8.5. For an affine morphism $\pi : X \to Y$, define the functors π^0 and π_0 gluing from affine pieces. In other words,

$$\pi^0(\mathcal{G})(\pi^{-1}(U)) := (\pi|_{\pi^{-1}(U)})^0(\mathcal{G}(U))$$

and

$$\pi_0(\mathcal{F})(U) := (\pi|_{\pi^{-1}(U)})_0(\mathcal{F}(\pi^{-1}(U))$$

for any open affine $U \subset Y$.

Example 8.6. Let $i_0 : Z \to X$ be a closed embedding of a smooth subvariety. One can choose local coordinates x_i, such that Z is given by $x_{m+1} = \cdots = x_n = 0$,

$$\mathcal{D}(X) \simeq \mathcal{O}(X) \otimes_{\mathbb{K}} \mathbb{K}[\partial_1, \ldots, \partial_n], \text{ and } \mathcal{O}(Z) \simeq \mathcal{O}(X)/J, \text{ where } J := \langle x_{m+1}, \ldots, x_n \rangle.$$

Then $i^0\mathcal{F} = \mathcal{F}/J$.

Exercise 8.7. Let $V \subset X$ be an open affine subset and let $i : V \hookrightarrow X$ denote the embedding. Then

(i) $i_0(\mathcal{F})(U) = \mathcal{F}(V \cap U)$ for any $\mathcal{F} \in \mathcal{M}(\mathcal{D}_V)$ and any open $U \subset X$.
(ii) The functor $i_0 : \mathcal{M}(\mathcal{D}_V) \to \mathcal{M}(\mathcal{D}_X)$ is right-adjoint to the restriction functor $Res_V : \mathcal{M}(\mathcal{D}_X) \to \mathcal{M}(\mathcal{D}_V)$.
(iii) The functors i_0 and Res_V are exact.

?? add on functoriality in general.

Fact 8.8. *For a coherent sheaf TFAE:*

(i) It is locally free
(ii) It is locally projective
(iii) The dimension of the fiber is locally constant.

For affine X, locally projectives are projectives. For non-affine X the categories of $\mathcal{O}_X$-modules and $\mathcal{D}_X$-modules do not have enough projectives, but:

Fact 8.9. $\mathcal{M}(\mathcal{O}_X)$ *and* $\mathcal{M}(\mathcal{D}_X)$ *have enough injectives.*

Proof. Let us show this for $\mathcal{D}_X$-modules, since the proof for $\mathcal{O}_X$-modules is identical.

First we prove for affine X. For a projective right $\mathcal{D}_X$-module P the module $\mathrm{Hom}_{\mathbb{K}}(P, \mathbb{K})$ is an injective left $\mathcal{D}_X$-module. For any projective P and an epimorphism $P \to \mathrm{Hom}_{\mathbb{K}}(M, \mathbb{K}) \to 0$ we have embeddings

$$M \hookrightarrow \mathrm{Hom}_{\mathbb{K}}(\mathrm{Hom}_{\mathbb{K}}(M, \mathbb{K}), \mathbb{K}) \hookrightarrow \mathrm{Hom}_{\mathbb{K}}(P, \mathbb{K}).$$

For non-affine varieties, choose a finite open affine cover $X = \bigcup_j U_j$, and consider $i_0 : \mathcal{M}(\mathcal{D}_{U_j}) \to \mathcal{M}(\mathcal{D}_X)$. The functor i_0 is exact and maps injective sheaves to injective ones. Since $\mathcal{F} \restriction_{U_j}$ embeds into injective Q_j, $\mathcal{F}$ embeds into $\bigoplus_j i_{j*} Q_j$. $\square$

Definition 8.10. A $\mathcal{D}_X$-module is called coherent if it is locally finitely generated.

Recall that for an affine variety X, $\mathrm{Gr}\,\mathcal{D}(X) = \mathcal{O}(T^*X)$.

Definition 8.11. For $\mathcal{F} \in \mathcal{M}_{\mathrm{coh}}(\mathcal{D}_X)$ choose a good filtration on $\mathcal{F}$, and define

$$\mathrm{Sing\,supp}_U\, F := \mathrm{supp}\,\mathrm{Gr}\,\mathcal{F}(U) \subset T^*X, \text{ and } \mathrm{Sing\,supp}\,\mathcal{F} := \bigcup_U \mathrm{Sing\,supp}_U\,\mathcal{F}.$$

By Lemma 4.3 and Corollary 4.5 this does not depend on the choice of a good filtration on $\mathcal{F}$.

Theorem 4.16 holds for singular support as well, though we won't prove it.

Theorem 8.12 (Kashiwara-Kawai-Sato, Gabber)**.** *For any* $\mathcal{F} \in \mathcal{M}_{\mathrm{coh}}(\mathcal{D}_X)$*,* $\mathrm{Sing\,supp}\,\mathcal{F}$ *is a coisotropic subvariety of* T^*X*.*

This implies the Bernstein inequality, namely $\dim \mathrm{Sing\,supp}\,\mathcal{F} \geq \dim X$ if $\mathcal{F} \neq 0$. Another way of proving the Bernstein inequality is to reduce it to affine varieties, then to affine spaces, then use Proposition 4.13 to reduce to the classical Bernstein inequality for the arithmetic filtration (Theorem 2.1 above).

However, we are going to give a direct proof of the Bernstein inequality in the next section.

9. Kashiwara's lemma and its corollaries

Let $Z \subset X$ be a closed smooth subvariety and let $i : Z \hookrightarrow X$ denote the embedding. Let $\mathcal{M}^{\mathrm{r}}_Z(\mathcal{D}_X)$ denote the category of right $\mathcal{D}_X$-modules supported at Z. Our goal in this section is to prove and use the following theorem.

Theorem 9.1 (Kashiwara)**.** *The functor i_0 is an equivalence $\mathcal{M}^{\mathrm{r}}(\mathcal{D}_Z) \simeq \mathcal{M}^{\mathrm{r}}_Z(\mathcal{D}_X)$.*

For the proof we will need some constructions and lemmas.

Definition 9.2. Define $i' : \mathcal{M}^{\mathrm{r}}(\mathcal{D}_X) \to \mathcal{M}^{\mathrm{r}}(\mathcal{D}_Z)$ by

$$i'(\mathcal{F}) := \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{D}_{Z\to X}, \mathcal{F}).$$

For $\mathcal{F} \in \mathcal{M}^{r}(\mathcal{D}_X)$ define $\Gamma_Z(\mathcal{F})(U) := \{\xi \in \mathcal{F}(U) \mid \operatorname{supp}\xi \subset Z\}$.

Exercise 9.3. (i) For affine X, $i'(M) \simeq \mathrm{Ann}_M I(Z)$.
(ii) $i' i_0 \mathcal{H} \simeq \mathcal{H}$ for any $\mathcal{H} \in \mathcal{M}(\mathcal{D}_Z)$.
(iii) i_0 is left adjoint to i'.

From the adjunction, we have a counit map $i_0 i' \mathcal{F} \to \mathcal{F}$.
Recall the following lemma

Lemma 9.4. *Let $M \in \mathcal{M}(\mathcal{D}_1)$. Assume $M = \bigcup_i \ker x^i$. Then $M = \bigoplus_i \partial^i \ker x$.*

Example 9.5. Distributions on $\mathbb{R}$ supported at 0 are sums of derivatives of the δ-function.

Lemma 9.6 (Standard in algebraic geometry)**.** *For any $x \in Z$, there is an open neighborhood $U \subset X$ and a quasi-coordinate system x_i on U, such that $Z \cap U$ is given by $x_{m+1} = \cdots = x_n = 0$, and the Jacobian $\det(\partial x_i)$ does not vanish.*

Theorem 9.7. *$\varphi : i_0 i' \mathcal{F} \to \Gamma_Z(\mathcal{F})$ is an isomorphism.*

Proof. It's enough to show this locally. Choose a quasi-coordinate system on X (or an open subset of it), such that $Z = \{x_{m+1} = \cdots = x_n = 0\}$. We can assume $n = m+1$ by induction (locally there is a flag of smooth subvarieties, constructed using x_i). The induction step will be for $Z \subset Y \subset X$:

$$(Y \to X)_0 (Z \to Y)_0 (Z \to Y)' (Y \to X)' \mathcal{F} \simeq (Y \to X)_0 \Gamma_Z (Y \to X)' \mathcal{F} \simeq$$
$$\simeq (Y \to X)_0 (Y \to X)' \Gamma_Z \mathcal{F} \simeq \Gamma_Y \Gamma_Z \mathcal{F} \simeq \Gamma_Z \mathcal{F}$$

The nontrivial isomorphism here follows by noting that Γ_Y consists of sections supported at Y, and $(Z \to Y)'$ consists of the sections killed by $I(Z)$.

Finally, define $Z := \{x_n = 0\}$. Let M be a right $\mathcal{D}(X)$-module, and $N := \Gamma_Z(M)$.

$$i' M = \ker(x_n \restriction_M)$$

and $i_0 i' M = \bigoplus_j i' M \, \partial^j_{x_n}$ because $\mathcal{D}_{Z\to X} \simeq \mathcal{O}_Z \otimes_{\mathcal{O}_X} \mathcal{D}_X \simeq \mathcal{D}_X / x_n \mathcal{D}_X \simeq \mathcal{D}_Z \otimes_{\mathbb{K}} \mathbb{K}[\partial_{x_n}]$. By Lemma 9.4,

$$\bigoplus_j \ker(x_n \restriction_N) \, \partial^j_{x_n} \simeq N$$

□

Corollary 9.8 (Kashiwara)**.** *The functors i_0 and i' define an equivalence of categories*

$$\mathcal{M}^{\mathrm{r}}(\mathcal{D}_Z) \simeq \mathcal{M}^{\mathrm{r}}_Z(\mathcal{D}_X).$$

9.1. **Corollaries.**

Lemma 9.9 (Exercise)**.** *For $\mathcal{H} \in \mathcal{M}_{\text{coh}}(\mathcal{D}_Z)$, $i_0\mathcal{H} \in \mathcal{M}_{\text{coh}}(\mathcal{D}_X)$ and*

$$\operatorname{Sing\,supp}(i_0\mathcal{H}) = \{(x,\xi) \in T^*X \mid (x, p_x\xi) \in \operatorname{Sing\,supp}\mathcal{H}\},$$

where $p_x : (T_x^X)^* \twoheadrightarrow (T_x^Z)^*$ is the dual map to the embedding $T_xZ \hookrightarrow T_xX$.

Corollary 9.10 (Bernstein's inequality)**.** *For any $\mathcal{F} \in \mathcal{M}_{\text{coh}}(\mathcal{D}_X)$,*

$$\dim \operatorname{Sing\,supp}\mathcal{F} \geq \dim X.$$

Proof. Let $\mathcal{F} \in \mathcal{M}^{\text{r}}_{\text{coh}}(\mathcal{D}_X)$. Suppose that $\dim \operatorname{Sing\,supp}\mathcal{F} < \dim X$.

Let $p_X : T^*X \to X$ be the canonical projection. Let $Z := \overline{(p_X(\operatorname{Sing\,supp}\mathcal{F}))} \subsetneq X$. Then $\dim Z < \dim X$. There is open dense $U \subset X$, such that $Z' := U \cap Z$ is nonsingular (and nonempty). $\mathcal{F}' := \mathcal{F}\restriction_U$. Then $\operatorname{supp}\mathcal{F}' \subset Z'$. By Kashiwara's lemma, $\mathcal{F}' \simeq i_0 i'\mathcal{F}'$, where $i : Z' \to U$. By induction hypothesis, $\dim \operatorname{Sing\,supp} i'\mathcal{F}' \geq \dim Z'$. Thus

$$\dim \operatorname{Sing\,supp} i_0 i'\mathcal{F}' \geq \dim \operatorname{Sing\,supp} i'\mathcal{F}' + \dim U - \dim Z' \geq \dim X.$$

But $\dim \operatorname{Sing\,supp}\mathcal{F}' < \dim X$ by assumption. This leads to a contradiction. $\square$

Lemma 9.11. *Let $\mathcal{F} \in \mathcal{M}_{\text{coh}}(\mathcal{D}_X)$. TFAE:*

(1) $\operatorname{Sing\,supp}\mathcal{F} \subset X \times \{0\} \subset T^*X$
(2) $\mathcal{F} \in \mathcal{M}_{\text{coh}}(\mathcal{O}_X)$
(3) $\mathcal{F}$ is locally free of finite rank over $\mathcal{O}_X$

Proof. $3 \Rightarrow 2$ is obvious, $2 \Rightarrow 1$ is obvious (just take the generators over $\mathcal{O}_X$ and use them to construct a good filtration).

$1 \Rightarrow 2$: Choose local coordinates in an open affine $U \subset X$. Let $M = \mathcal{F}(U)$. Choose generators $v_1, \ldots, v_n$ of M over $\mathcal{D}_U$. Then we assume that $Z(\sigma \operatorname{ann}\{v_i\}) = U \times \{0\}$ (where σ is the symbols, and Z is the variety of zeros). Then for any i, j there is ℓ_{ij}, such that

$$\partial_j^{\ell_{ij}} v_i \in \mathcal{D}_U^{<\ell_{ij}}\{v_1, \ldots, v_n\}$$

Let $S := \left\{\partial_1^{\ell_1} \ldots \partial_m^{\ell_m} v_i \;\middle|\; \ell_j < \ell_{ij}\right\}$. Then this set generates $F(U)$ over $\mathcal{O}_U$, so F is coherent over $\mathcal{O}_X$.

$2 \Rightarrow 3$: We can assume that X is affine. Let $\ell := \min_x \dim \mathcal{F}_x$. Then there is some open $U \subset X$, such that $\dim \mathcal{F}_x = \ell$ for all $x \in U$ (we assume that X is connected and irreducible). Suppose by way of contradiction that there exists $x \in X$, such that $\dim F_x > \ell$. Then there is a smooth affine curve $\nu : C \to X$ passing through x (cf. curve selection lemma), such that all the other points of this curve are in U. Take the $\mathcal{D}_C$-module $\nu^0 F$. It's $\mathcal{O}_C$-coherent because it's the pullback of $\mathcal{O}$-modules, and this operation preserves coherence. On the other hand, $\mathcal{O}_C$ is a Dedekind domain, so since $M := \nu^0 F$ is not locally free, it must have torsion. The torsion part M^{tor} is also a $\mathcal{D}_C$-module, and it has finite support $i : Z \subset C$. Thus $M^{\text{tor}} = i_0 V$ for some $\mathcal{D}_Z$-module V. But $i_0 V$ is not finitely generated over $\mathcal{O}_C$ unless $V = 0$. $\square$

Definition 9.12. $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules are called smooth.

Corollary 9.13. *Let $\mathcal{F}$ be a holonomic $\mathcal{D}_X$-module. Then there exists an open dense $U \subset X$, such that $\mathcal{F}\restriction_U$ is smooth (possibly trivial).*

Proof. $\operatorname{Sing\,supp} \mathcal{F}$ is n-dimensional, so it consists of a part of the form $U \times \{0\}$ and something else that projects to a lower-dimensional subvariety of X. □

Definition 9.14. For a closed subvariety $X \subset \mathbb{A}^n$ define the category of $\mathcal{D}_X$-modules as the category of $\mathcal{D}_{\mathbb{A}^n}$-modules supported at X.

Theorem 9.15. *This definition doesn't depend on the embedding.*

Proof. Let $\nu : X \to \mathbb{A}^n$, $\mu : X \to \mathbb{A}^m$. Take the embedding $\nu \times \mu : X \to \mathbb{A}^{n+m}$. Then there is $\rho : \mathbb{A}^n \to \mathbb{A}^m$, such that $\mu = \rho\nu$. Thus we have a closed embedding $i := \mathrm{id} \times \rho : \mathbb{A}^n \to \mathbb{A}^{n+m}$. Then $\mathcal{M}_X(\mathcal{D}_{\mathbb{A}^n}) \overset{i_0}{\simeq} \mathcal{M}_X(\mathcal{D}_{\mathbb{A}^{n+m}})$. □

Definition 9.16. Define $\mathcal{D}$-modules on general varieties by gluing affine ones. Note that for affine ones the notion is local.

For $F \in \mathcal{M}_X(\mathcal{D}_{\mathbb{A}^n})$ define

$$\operatorname{Sing\,supp}_X(F) := p_X(\operatorname{Sing\,supp} F),$$

where $p_X : T^*\mathbb{A}^n \restriction_X \to T^*X$.

10. $\mathcal{D}$-modules on the projective space

$$V \overset{j}{\supset} V^\times \xrightarrow{p} \mathbb{P}(V)$$

For an $\mathcal{O}_{\mathbb{P}(V)}$-module $\mathcal{F}$ define

$$p^*\mathcal{F} := \mathcal{O}_{V^\times} \otimes_{\mathcal{O}_{\mathbb{P}(V)}} \mathcal{F}, \quad \tilde{\mathcal{F}} := j_*p^*\mathcal{F}.$$

There is an action of $\mathbb{G}_m$ on $\tilde{\mathcal{F}}$ by dilation, so it defines a grading on the global sections $\Gamma\left(\tilde{\mathcal{F}}\right)$, and $\Gamma(\mathcal{F}) = \left(\Gamma\left(\tilde{\mathcal{F}}\right)\right)^0$.

Let $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ be an exact sequence of $\mathcal{D}_{\mathbb{P}(V)}$-modules. Then the sequence

$$p^0\mathcal{F}_1 \to p^0\mathcal{F}_2 \to p^0\mathcal{F}_3$$

is exact. While $\tilde{\mathcal{F}}_1 \to \tilde{\mathcal{F}}_2 \to \tilde{\mathcal{F}}_3$ may not be exact, the homology $\mathcal{H}$ is supported at 0. Thus $\mathcal{H} \simeq i_0 L$, where L is a vector space, and $i : \{0\} \hookrightarrow V$.

Let $E := \sum x_i\partial_i \in \mathcal{D}(V)$ be the Euler operator.

Exercise 10.1. On $\Gamma(j_0L)$, E has negative eigenvalues only.

$(\Gamma(j_0L))^0 = 0$, thus $(\Gamma(\mathcal{H}))^0 = 0$, so

$$\left(\Gamma\left(\tilde{\mathcal{F}}_1\right)\right)^0 \to \left(\Gamma\left(\tilde{\mathcal{F}}_2\right)\right)^0 \to \left(\Gamma\left(\tilde{\mathcal{F}}_3\right)\right)^0$$

is exact. On the other hand, $\left(\Gamma\tilde{\mathcal{F}}_i\right)^0 \simeq \Gamma\mathcal{F}_i$. Thus:

Lemma 10.2. *The functor of global sections*

$$\Gamma_{\mathbb{P}(V)} : \mathcal{M}(\mathcal{D}_{\mathbb{P}^n}) \to \mathcal{M}(\Gamma(\mathcal{D}_{\mathbb{P}^n}))$$

is exact.

Exercise 10.3. $\mathcal{D}_{\mathbb{P}^n} \simeq \mathcal{D}_n^0/\mathcal{D}_n^0 E$, where $\mathcal{D}_n^0$ is the zero-part of the grading on $\mathcal{D}_n$ given by the commutator with the Euler vector field. In other words, $\deg x_i = 1$, $\deg \partial_i = -1$.

Exercise 10.4. For any graded $\mathbb{K}[x_0, \ldots, x_n]$-module we define a quasicoherent sheaf on the projective space M' by $M'(U) := \left(M\left(P^{-1}(U)\right)\right)^0$, where $P : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is the canonical projection. Any quasicoherent sheaf on $\mathbb{P}^n$ is obtained this way. More precisely,

$$\mathcal{M}^{\mathrm{qc}}(\mathcal{O}_{\mathbb{P}^n}) \simeq \mathcal{M}^{\mathrm{qc}}(\mathcal{O}_{\mathbb{A}^n}) / \mathcal{M}^{\mathrm{qc}}_{\{0\}}(\mathcal{O}_{\mathbb{A}^n})$$

(quotient w.r.t. a Serre subcategory).

Hint. $\mathbb{K}[x_0, \ldots, x_n] \simeq \bigoplus_{d \geq 0} \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$, where $\mathcal{O}_{\mathbb{P}^n}(d)$ is the sheaf on $\mathbb{P}^n$ obtained by shifting by d the grading in the graded module $\mathbb{K}[x_0, \ldots, x_n]$ (alternative description: $\mathcal{O}_{\mathbb{P}^n}(-1)$ is the canonical line bundle, $\mathcal{O}_{\mathbb{P}^n}(1)$ is its dual, and $\mathcal{O}_{\mathbb{P}^n}(d_1 + d_2) \simeq \mathcal{O}_{\mathbb{P}^n}(d_1) \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(d_2)$).

Take a sheaf $\mathcal{F}$, and the module $M_{\mathcal{F}} := \bigoplus_{d \geq 0} \Gamma(\mathbb{P}^n, \mathcal{F}(d))$, where $\mathcal{F}(d) := \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(d)$. Now take the sheaf $M'_{\mathcal{F}}$ corresponding to M_F. We claim that $M'_{\mathcal{F}} \simeq \mathcal{F}$. After that we prove that the kernel of the functor $M \mapsto \mathcal{F}$ consists of the sheaves supported at 0. □

Lemma 10.5. $\Gamma_{\mathbb{P}(V)} : \mathcal{M}(\mathcal{D}_{\mathbb{P}^n}) \to \mathcal{M}(\Gamma \mathcal{D}_{\mathbb{P}^n})$ *is faithful.*

Proof. Since $\Gamma_{\mathbb{P}(V)}$ is exact, it is enough to show that $\Gamma(\mathcal{F}) \neq 0$ for $\mathcal{F} \neq 0$.

Let j be such that $\operatorname{supp} M^j \not\subset \{0\}$ and $\operatorname{supp} M^\ell \subset \{0\}$ $\forall l$ with $|\ell| < j$. We want to show that $j = 0$. Suppose first that $j < 0$ and let $\xi \in M^j$, such that $\operatorname{supp} \xi \not\subset \{0\}$. Then there is $0 \leq i \leq n$, such that $\operatorname{supp} x_i \xi \not\subset \{0\}$. But $x_i \xi \in M^{j+1}$, so this contradicts our assumption. Similarly, for $j > 0$, take $\xi \in M^j$. $j\xi = E\xi = \sum_i x_i \partial_i \xi$, so there is i, such that $\operatorname{supp} \partial_i \xi \not\subset \{0\}$. But $\partial_i \xi \in M^{j-1}$, so again we get a contradiction. □

Lemma 10.6. $\operatorname{Hom}(\mathcal{D}_{\mathbb{P}^n}, \mathcal{F}) \simeq \Gamma(\mathcal{F})$.

Proof. The internal Hom is $\mathcal{F}$, so the categorical Hom consists of its global sections. □

Corollary 10.7 (Bernstein-Beilinson, ??)**.** $\mathcal{D}_{\mathbb{P}^n}$ *is a projective generator of* $\mathcal{M}(\mathcal{D}_{\mathbb{P}^n})$, *and thus* $\Gamma : \mathcal{M}(\mathcal{D}_{\mathbb{P}^n}) \to \mathcal{M}(\Gamma(\mathcal{D}_{\mathbb{P}^n}))$ *is an equivalence of categories.*

Theorem 10.8 (Bernstein-Beilinson, ??)**.** $\Gamma(\mathcal{D}_{\mathbb{P}^n}) = \mathcal{D}_{\mathbb{P}^n}(\mathbb{P}^n) \simeq \mathcal{D}_{n+1}^0 / E\mathcal{D}_{n+1}^0$, *where* $\mathcal{D}_{n+1}^0$ *is according to the grading* $\deg x_i = 1, \deg \partial_i = -1$, *and* E *is the Euler field.*

For the proof we will need some lemmas.

Exercise 10.9. There is a natural map $\mathcal{D}_{n+1}^0 / E\mathcal{D}_{n+1}^0 \to \mathcal{D}_{\mathbb{P}^n}(\mathbb{P}^n)$.

Exercise 10.10. $\operatorname{Gr}\left(\mathcal{D}_{n+1}^0 / E\mathcal{D}_{n+1}^0\right) \simeq \mathcal{O}_{T^*\mathbb{P}^n}(T^*\mathbb{P}^n)$.

Lemma 10.11. *For all smooth* X, $\operatorname{Gr} \mathcal{D}_X(X) \hookrightarrow \mathcal{O}_{T^*X}(T^*X)$.

Proof. $0 \to \mathcal{D}_X^{i-1} \to \mathcal{D}_X^i \to \operatorname{Sym}^i_{\mathcal{O}_X} \tau_X \to 0$,

so $0 \to \Gamma\mathcal{D}_X^{i-1} \to \Gamma\mathcal{D}_X^i \to \Gamma \operatorname{Sym}^i_{\mathcal{O}_X} \tau_X$.

On the other hand, $\bigoplus_i \operatorname{Sym}^i_{\mathcal{O}_X} \tau_X \simeq \mathcal{O}_{T^*X}$. □

Proof of the Theorem. $\varphi : \mathcal{D}_{n+1}^0 / E\mathcal{D}_n^0 \to \mathcal{D}_{\mathbb{P}^n}(\mathbb{P}^n)$. It's enough to show that $\operatorname{Gr}\varphi$ is an isomorphism. Now, $\operatorname{Gr}(\mathcal{D}_{n+1}^0 / E\mathcal{D}_n^0) \simeq \mathcal{O}_{T^*\mathbb{P}^n}(T^*\mathbb{P}^n)$, and $\operatorname{Gr}(\mathcal{D}_{\mathbb{P}^n}(\mathbb{P}^n))$ is embedded into $\mathcal{O}_{T^*\mathbb{P}^n}(T^*\mathbb{P}^n)$. This embedding is compatible with $\operatorname{Gr}(\varphi)$, and thus both the embedding and $\operatorname{Gr}(\varphi)$ are isomorphisms. □

10.1. **Twisted differential operators on the projective space.** In Definition 7.6 we defined the algebra of differential operators on any module over the algebra of polynomials on an affine variety. Later we showed that this definition commutes with localization by polynomials. This gives the definition of the sheaf of algebras of differential operators on a coherent sheaf over any algebraic variety X. The obtained algebra is well-behaved only if the sheaf is locally free. If the sheaf is invertible (i.e. is a line bundle), this sheaf of algebras is locally isomorphic to $\mathcal{D}_{\mathbb{P}^n}$.

Definition 10.12. A sheaf of twisted differential operators on a (smooth, separated) algebraic variety X is a sheaf of $\mathcal{O}_X$-algebras that is locally isomorphic to $\mathcal{D}_X$ (in short a TDO on X).

Let us consider the case $X = \mathbb{P}^n$. Any invertible sheaf on $\mathbb{P}^n$ is isomorphic to $\mathcal{O}(s)$ for some $s \in \mathbb{Z}$. One can define $\mathcal{O}(s)$ to correspond the construction in Exercise 10.4 to the graded module $M = \mathbb{K}[x_0, \dots, x_n]$ with grading shifted by d. Another way to define an invertible sheaf $\mathcal{F}$ is describe the automorphism of $\mathcal{O}(U_i \cap U_j)$ given by the identifications $\mathcal{F}(U_i) \simeq \mathcal{O}_X(U_i)$ for some open affine cover $\{U_i\}$ of X on which $\mathcal{F}$ trivializes. For $\mathcal{O}(s)$ we can choose the standard cover $U_i := \{x_i \neq 0\} \cong \mathbb{A}^n$ of $\mathbb{P}^n$, on which the automorphisms are given by multiplication by $(x_i/x_j)^s$.

Let us describe $\mathcal{O}(s)$ by coordinate changes. We have to compute what happens to ∂_k when we twist it by $(x_i/x_j)^s$.

$$(x_i/x_j)^{-s} \cdot \partial_k \cdot (x_i/x_j)^s = \partial_k + s\partial_k(x_i/x_j) \cdot (x_i/x_j)^{-1}, \tag{1}$$

Since $\mathcal{D}_n$ is generated as a $\mathbb{K}[x_1, \dots x_n]$-algebra by $\xi_1, \dots, \xi_n$, this formula defines a sheaf of twisted differential operators on $\mathbb{P}^n$. In fact, we could put any scalar $\lambda \in \mathbb{K}$ instead of s in (1) and obtain a TDO on $\mathbb{P}^n$.

Exercise 10.13. Any TDO on $\mathbb{P}^n$ is given by the coordinate changes

$$\varphi_{ij}(\partial_k) := \partial_k + \lambda\partial_k(x_i/x_j) \cdot (x_i/x_j)^{-1}.$$

Exercise 10.14. Denote by $\mathcal{D}_{\mathbb{P}^n}(s)$ the sheaf of differential operators on $\mathcal{O}_{\mathbb{P}^n}(s)$. Then the global sections functor $\Gamma : \mathcal{M}(\mathcal{D}_{\mathbb{P}^n}(s)) \to \mathcal{M}(\Gamma(\mathcal{D}_{\mathbb{P}^n}(s)))$ is exact for $s > -n$ and faithful for $s \geq 0$.

Let us find a formula for obtaining TDOs from invertible sheaves on arbitrary (smooth, separated) varieties. Recall that a 1-form on an affine variety X is an $\mathcal{O}(X)$-module morphism $\tau_X \to \mathcal{O}(X)$. A 1-form λ is called closed if its differential $d\lambda$ vanishes. The differential can be defined as the two-form given by

$$\lambda(\xi, \eta) := \xi(\lambda(\eta) - \lambda(\xi(\eta)) - \lambda([\xi, \eta]), \quad \forall \xi, \eta \in \tau_X.$$

Exercise 10.15. Let X be affine. For a closed 1-form λ on X and $\eta \in \tau_X$ define $\varphi_\lambda(\eta) := \eta + \lambda(\eta) \in \mathcal{D}(X)$. Then φ_λ extends (uniquely) to an automorphism of $\mathcal{D}(X)$ as an $\mathcal{O}(X)$-algebra. Moreover, all automorphisms of $\mathcal{D}(X)$ as an $\mathcal{O}(X)$-algebra are obtained in this way.

For non-affine X, this exercise and the Chech cohomology yield that the TDOs on X are described by $\mathrm{H}^1(X, \Omega^1_{cl})$, where Ω^1_{cl} is the sheaf of closed 1-forms on X. The group of invertible sheaves on X (a.k.a. the Picard group) is isomorphic to $\mathrm{H}^1(X, \mathcal{O}_X^\times)$, where

$\mathcal{O}_X^\times$ is the sheaf of invertible regular functions on X. The logarithmic derivative gives a morphism of sheaves of abelian groups $\mathcal{O}_X^\times \to \Omega^1_{cl}$, which in turn gives a group homomorphism $\mathrm{H}^1(X, \mathcal{O}_X^\times) \to \mathrm{H}^1(X, \Omega^1_{cl})$. This homomorphism describes the correspondence between invertible sheaves and TDOs. For $X = \mathbb{P}^n$ we have $\mathrm{H}^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^\times) = \mathbb{Z}$ and $\mathrm{H}^1(\mathbb{P}^n, \Omega^1_{cl}) = \mathbb{K}$. Thus Exercise 10.15 generalizes (1).

11. The Bernstein-Kashiwara theorem on distributional solutions of holonomic modules

Let X be a smooth algebraic variety defined over $\mathbb{R}$, and let $\mathcal{S}_X^*$ denote the $\mathcal{D}_X$-module of tempered distributions on X. More precisely, for every open $U \subset X$ we take $\mathcal{S}_X^*(U) := \mathcal{S}^*(U(\mathbb{R}))$, the space of continuous functionals on the Fréchet space of Schwartz functions on $U(\mathbb{R})$. Let $\mathcal{M}_{\mathrm{hol}}(\mathcal{D}_X)$ denote the category of holonomic $\mathcal{D}_X$-modules. Our goal in this section is to prove and use the following theorem.

Theorem 11.1 (Bernstein-Kashiwara)**.** *Let $\mathcal{F} \in \mathcal{M}_{\mathrm{hol}}(\mathcal{D}_X)$. Then*

$$\dim \mathrm{Hom}\left(\mathcal{F}, \mathcal{S}_X^*\right) < \infty.$$

Lemma 11.2 (Exercise)**.** *Let $j : Z \subset X$ be a closed embedding of smooth affine algebraic varieties defined over $\mathbb{R}$. Then $\mathcal{S}^*(Z) \simeq j^! \mathcal{S}^*(X)$.*

Corollary 11.3. *It is enough to prove Theorem 11.1 for the case when X is an affine space.*

Proof. Let $X = \bigcup_{i=1}^r U_i$ be an open affine cover. Then

$$\mathrm{Hom}\left(\mathcal{F}, \mathcal{S}_X^*\right) \hookrightarrow \prod_i \mathrm{Hom}\left(\mathcal{F}(U_i), \mathcal{S}^*(U_i)\right)$$

by restriction. Let $\tau_i : U_i \to \mathbb{A}^{n_i}$ be closed embeddings. Then $\mathcal{S}^*(U_i) \simeq \tau_i^! \mathcal{S}^*(\mathbb{A}^{n_i})$. Hence by the adjunction,

$$\mathrm{Hom}\left(\mathcal{F}(U_i), \mathcal{S}^*(U_i)\right) \simeq \mathrm{Hom}\left(\mathcal{F}(U_i), \tau_i^! \mathcal{S}^*(\mathbb{A}^{n_i})\right) \simeq$$
$$\simeq \mathrm{Hom}\left((\tau_i)_0 \mathcal{F}(U_i), \mathcal{S}^*(\mathbb{R}^{n_i})\right)$$

Recall that the pushforward preserves holonomicity. ☐

From now on let $X = V := \mathbb{R}^n$ and M be a holonomic $\mathcal{D}_n$-module.

Definition 11.4. Let ω be the standard symplectic form on $V \oplus V^*$. Denote by $p_V : V \oplus V^* \to V$ and $p_{V^*} : V \oplus V^* \to V^*$ the natural projections. Define an action of the symplectic group $\mathrm{Sp}(V \oplus V^*, \omega)$ on the algebra $\mathcal{D}(V)$ by

$$(\partial_v)^g := \pi(g)(\partial_v) := p_{V^*}(g(v,0)) + \partial_{p_V(g(v,0))}, \quad w^g := \pi(g)w := p_{V^*}(g(0,w)) + \partial_{p_V(g(0,w))}$$

where $v \in V$, $w \in V^*$, ∂_v denotes the derivative in the direction of v, and elements of V^* are viewed as linear polynomials and thus differential operators of order zero. For a $\mathcal{D}(V)$-module M and an element $g \in \mathrm{Sp}(V \oplus V^*)$, we will denote by M^g the $D(V)$-module obtained by twisting the action of $D(V)$ by $\pi(g)$.

Since the above action of $\mathrm{Sp}(V \oplus V^*)$ preserves the Bernstein filtration on $\mathcal{D}(V)$, the following lemma holds.

Lemma 11.5. *For $M \in \mathcal{M}^f(\mathcal{D}(V))$ and $g \in \mathrm{Sp}(V \oplus V^*)$ we have $\mathrm{AV}(M^g) = g\mathrm{AV}(M)$.*

Lemma 11.6. *For any $g \in \mathrm{Sp}(V \oplus V^*)$, $\mathcal{S}(V)^g \simeq \mathcal{S}(V)$, and thus $\mathcal{S}^*(V)^g \simeq \mathcal{S}^*(V)$.*

We will prove this lemma in §11.1.

Lemma 11.7. *Let $C \subset V \oplus V^*$ be a closed conic subvariety of dimension n. Then there exists a Lagrangian subspace $W \subset V \oplus V^*$, such that the projection of C onto $(V \oplus V^*)/W$ is a finite map.*

Proof. First we prove that there is a Lagrangian subspace L, such that $L \cap C = \{0\}$. For that $\mathcal{L}$ denote the variety of Lagrangian subspaces and consider

$$Y := \{(\alpha, \beta) \in P(C) \times \mathcal{L} \mid \alpha \subset \beta\},$$

where $P(C)$ is the space of lines inside C (i.e. the projectivization). We have maps $q : Y \to P(C)$ and $q' : Y \to \mathcal{L}$, and we need to show that q' is not onto. For this it's enough to show that $\dim Y < \dim \mathcal{L}$. Now we see that

$$\dim \mathcal{L} = \frac{1}{2}n(n+1), \text{ and } \dim Y = \dim q(Y) + \dim q^{-1}(x),$$

where $q^{-1}(x)$ is a generic fiber. Now, for every line $\ell \subset L$ we have $L \subset \ell^{\perp}$ =:the orthogonal complement to ℓ w.r. to the symplectic form. Thus L/ℓ is a Lagrangian subspace of $\ell^{\perp}/\ell$, and thus the dimension of $q^{-1}(x)$ is at most $(n-1)n/2 = n(n+1)/2 - n$. Thus

$$\dim Y \leq \frac{1}{2}(n-1)n + \dim P(C) = \frac{1}{2}n(n+1) - 1 < \dim \mathcal{L}.$$

Now we prove the following fact: over $\mathbb{C}$, if $W \subset U$ are vector spaces, and $C \subset U$ is a conic subvariety, such that $C \cap W = \{0\}$, then the projection $C \to U/W$ is finite. By induction on dimension we can reduce to the case $\dim W = 1$ (if it's true for $\dim W = l$ then take iterated projections, first w.r.t. W, then w.r.t. a larger subspace).

Let p be a homogeneous polynomial vanishing on C but not on W. Then

$$p(x_1, \dots, x_n) = \sum_{i=1}^{N} p_i(x_1, \dots, x_{n-1})\, x_n^i,$$

where x_i are linear coordinates, s.t. $W = \{x_1 = \dots = x_{n-1} = 0\}$. Thus $x_n \restriction_C$ satisfies a monic polynomial over $\mathcal{O}(U/W)$. Indeed, the leading term p_N is constant — otherwise this leading term would vanish on W, so by homogeneity we would have $\deg_{x_1,\dots,x_{n-1}} p_i > 0$ for all i, so p would vanish on W. Now since on $\mathcal{O}(C)$ the element x_n satisfies a monic polynomial over $\mathcal{O}(U/W)$, the ring extension $\mathcal{O}(U/W) \to \mathcal{O}(C)$ is integral, so the map $C \to U/W$ is finite.

Tanking $U := V \oplus V^*$ and $W := L$, such that the projection of C onto $(V \oplus V^*)/W$ is a finite map. $\square$

Corollary 11.8. *For any $M \in \mathcal{M}_{hol}(\mathcal{D}_V)$ there exists $g \in \mathrm{Sp}(V \oplus V^*)$ such that M^g is smooth.*

Proof. Since M is holonomic, we have $\dim \mathrm{AV}(M) = n$. Thus, by the lemma, there exists a Lagrangian subspace $W \subset V \oplus V^*$, such that the projection of $\mathrm{AV}(M)$ onto $(V \oplus V^*)/W$ is a finite map. Since $\mathrm{Sp}(V \oplus V^*)$ acts transitively on the variety of Lagrangian subspaces, there exists $g \in \mathrm{Sp}(V \oplus V^*)$ such that $g^{-1}(W) = V^*$, and thus $g\mathrm{AV}(M)$ is finite over V. By Lemma 11.5, $\mathrm{AV}(M^g) = g(\mathrm{AV}(M))$. Thus M^g is finitely-generated over $\mathcal{O}(V)$ and thus smooth. □

Lemma 11.9. *Let M be a smooth $D(\mathbb{C}^n)$-module of rank r. Embed the space $An(\mathbb{C}^n)$ of analytic functions on $\mathbb{C}^n$ into $\mathcal{D}^*(\mathbb{R}^n)$ using the Lebesgue measure. Then*

$$\mathrm{Hom}(M, \mathcal{D}^*(\mathbb{R}^n)) = \mathrm{Hom}(M, An(\mathbb{C}^n)) \text{ and } \dim \mathrm{Hom}(M, \mathcal{D}^*(\mathbb{R}^n)) = \mathrm{rank}\, M,$$

where $\mathrm{rank}\, M$ *is the rank of M as a vector bundle.*

Proof. Let $M_{An} := M \otimes_{\mathcal{O}(\mathbb{C}^n)} An(\mathbb{C}^n)$ and $\mathcal{D}_{An}(\mathbb{C}^n) := \mathcal{D}_n \otimes_{\mathcal{O}(\mathbb{C}^n)} An(\mathbb{C}^n)$ be the analytizations of M and $\mathcal{D}_n$. Then

$$\mathrm{Hom}_{\mathcal{D}_n}(M, \mathcal{D}^*(\mathbb{R}^n)) \cong \mathrm{Hom}_{D_{An}(\mathbb{C}^n)}(M_{An}, \mathcal{D}^*(\mathbb{R}^n)).$$

Since M_{An} is also smooth, $M_{An} \cong An(\mathbb{C}^n)^r$. Thus it is left to prove that

$$\mathrm{Hom}_{D_{An}(\mathbb{C}^n)}(An(\mathbb{C}^n), \mathcal{D}^*(\mathbb{R}^n)) = \mathrm{Hom}_{D_{An}(\mathbb{C}^n)}(An(\mathbb{C}^n), An(\mathbb{C}^n))$$

and the latter space is one-dimensional. This follows from the fact that a distribution with vanishing partial derivatives is a multiple of the Lebesgue measure. □

Corollary 11.10. *If a distribution generates a smooth $\mathcal{D}$-module then the distribution is an analytic measure.*

Proof of Theorem 11.1. By Corollary 11.3 we can assume that $X = V = \mathbb{R}^n$. By Corollary 11.8 there exists $g \in \mathrm{Sp}(V \oplus V^*)$ such that $\mathcal{F}^g$ is smooth. By Lemma 11.6 we have

$$\mathrm{Hom}\left(M, \mathcal{S}^*\left(V\right)\right) \simeq \mathrm{Hom}\left(M^g, \left(\mathcal{S}^*\left(V\right)\right)^g\right) \simeq \mathrm{Hom}\left(M^g, \mathcal{S}^*\left(V\right)\right).$$

Finally, $\dim \mathrm{Hom}\left(M^g, \mathcal{S}^*\left(V\right)\right) < \infty$ by Lemma 11.9. □

Let an algebraic group G act algebraically on a smooth algebraic variety X, both defined over $\mathbb{R}$.

Corollary 11.11. *If G has finitely many orbits on X then* $\dim\left(\mathcal{S}^*\left(M\right)\right)^G < \infty$.

Proof. The Lie algebra $\mathfrak{g}$ acts on X by vector fields $\xi_\alpha, \alpha \in \mathfrak{g}$. Define a $\mathcal{D}_X$-module $\mathcal{F}$ on X by $\mathcal{F}\left(U\right) := \mathcal{D}_X\left(U\right)/\mathcal{D}_X\left(U\right)\{\xi_\alpha \restriction_U\}$. Then the solutions of this $\mathcal{D}$-module with values in $\mathcal{S}_X^*$ are exactly the G-invariant distributions. Now modulo the previous result, it remains to show that $\mathcal{F}$ is holonomic. By construction we have

$$\mathrm{Sing\,supp}\, F \subset \{(x, \varphi) \in T^*M \mid \forall \alpha \in \mathfrak{g} : \langle \varphi, \xi_\alpha\left(x\right)\rangle = 0\} = \bigcup_x \mathrm{CN}_{Gx}^X$$

where CN_{Gx}^X is the conormal bundle of the orbit Gx. Since there are finitely many orbits, this is a finite union. All conormal bundles have dimension $\dim X$, so the same is true for their finite union. □

A bit more careful argument actually proves a bit stronger statement.

Theorem 11.12 (Aizenbud-Gourevitch-Minchenko)**.** *If G has finitely many orbits on X and $\mathcal{E}$ is an algebraic G-equivariant bundle on X then for any $n \in \mathbb{N}$ there is $C_n \in \mathbb{N}$, such that for any n-dimensional $\mathfrak{g}$-module τ,*

$$\dim \mathrm{Hom}_{\mathfrak{g}}\left(\tau, \mathcal{S}^*\left(X, \mathcal{E}\right)\right) \leq C_n.$$

Exercise 11.13. Let $\mathbb{R}$ act on $\mathbb{R}P^1$ by shifts. Compute the dimension of $\left(\mathcal{S}^*\left(\mathbb{R}P^1\right)\right)^{\mathbb{R}}$.

This exercise does not use the technique of this section, but rather demonstrates the nature of the question considered in the last theorem.

Solution. We have an open invariant subset $\mathbb{R} \subset \mathbb{R}P^1$, and the restriction of any invariant distribution to this open subset is a scalar multiple of the Lebesgue measure dx on $\mathbb{R}$. Let us now analyze another open set - the complement to $\{0\}$. Let us bring infinity to the point 0 by the coordinate change $x \mapsto t = 1/x$. Then $\partial_x = -t^2\partial_t$. Thus, on this open set we have a 2-dimensional space of invariant distributions, spanned by δ_0, and δ_0'. We also see that the restriction of dx to the intersection of the two open sets is not invariant, and thus dx does not extend to an invariant distribution on $\mathbb{R}P^1$.

On the other hand, δ_0 and δ_0' do extend by zero to all of $\mathbb{R}P^1$, and thus the space in question is 2-dimensional. □

Remark 11.14. We used that the algebraic group G has finitely many orbits on the algebraic variety X. This is equivalent to $G(\mathbb{C})$ having finitely many orbits on $X(\mathbb{C})$, but **not** equivalent to $G(\mathbb{R})$ having finitely many orbits on $X(\mathbb{R})$. Taito Tauchi ([Tau18]) constructed an example in which $G(\mathbb{R})$ has finitely many orbits on $X(\mathbb{R})$, but G has infinitely many orbits on X and the space $\mathcal{S}^*(X(\mathbb{R}))^{G(\mathbb{R})}$ is infinite-dimensional. Thus, the theory of D-modules, or at least some algebraic geometry, is required to prove Theorem 11.12.

11.1. **Proof of Lemma 11.6.** This section requires some knowledge of representation theory.

Definition 11.15. Let $V := \mathbb{R}^n$ and let ω be the standard symplectic form on $W_n := V \oplus V^*$. The Heisenberg group H_n is the algebraic group with underlying algebraic variety $W_n \times \mathbb{R}$ with the group law given by

$$(w_1, z_1)(w_2, z_2) = (w_1 + w_2, z_1 + z_2 + 1/2\omega(w_1, w_2)).$$

Define a unitary character χ of $\mathbb{R}$ by $\chi(z) := \exp(2\pi i z)$.

Definition 11.16. The oscillator representation of H_n is given on the space $L^2(V)$ by

$$(\sigma(x, \varphi, z)f)(y) := \chi(\varphi(y) + z))f(x + y). \tag{2}$$

Note that the center of H_n is $0 \times \mathbb{R}$, and it acts on σ by the character χ, which can be trivially extended to a character of $V^* \times \mathbb{R}$.

It is easy to see that σ is the unitary induction of (the extension of) the character χ from $V^* \times \mathbb{R}$ to $H_n = (V \oplus V^*) \times \mathbb{R}$.

Lemma 11.17. *The space of smooth vectors in σ is $\mathcal{S}(V)$, and the Lie algebra of H_n acts on it by*

$$\sigma(v)f := \partial_v f,\ \sigma(\varphi)f := \varphi f,\ \sigma(z)f := 2\pi i z f. \tag{3}$$

Proof. Formula (3) is obtained from (2) by derivation. Now, it is known that the space of smooth vectors in a unitary induction consists of the smooth L^2 functions whose derivatives also lie in L^2. □

Theorem 11.18 (Stone-von-Neumann)**.** *The oscillator representation σ is the only irreducible unitary representation of H_n with central character χ.*

Idea of the proof. Let me ignore all the analytic difficulties. Consider the normal commutative subgroup $A := V \times \mathbb{R}$. Conjugation in H_n defines an action of V on the dual group of A. This action has only two orbits. The closed orbit is the singalton $\{1\}$ and the open orbit $\mathcal{O}$ is the complement to the closed one. The restriction $\sigma|_A$ decomposes to a direct integral of characters in $\mathcal{O}$, each "with multiplicity one". The restriction of any non-zero subrepresentation $\rho \subset \sigma$ to A will also include χ, and thus the whole orbit $\mathcal{O}$ of χ. Thus $\rho = \sigma$ and σ is irreducible.

Now let τ be any irreducible unitary representation of H_n with central character χ. Then the restriction of τ to A will again include all the characters in $\mathcal{O}$ with multiplicity one. Thus τ is the induction of an irreducible representation of the stabilizer of χ in H_n. However, this stabilizer is A and thus $\tau \simeq \sigma$. □

Note that the symplectic group $\mathrm{Sp}(V \oplus V^*)$ acts on H_n by automorphisms, preserving the center. Thus the theorem implies the following corollary.

Corollary 11.19. *For every $g \in \mathrm{Sp}(V \oplus V^*)$ there exists a (unique up to a scalar multiple) linear automorphism T of $\mathcal{S}(V)$ such that for any $h \in H_n$ we have $\sigma(h^g) = T\sigma(h)T^{-1}$.*

Since the Lie algebra of H_n generates $\mathcal{D}_n$, this corollary implies Lemma 11.6.

Remark 11.20. The uniqueness part of Corollary 11.19 follows from Schur's lemmas. Corollary 11.19 defines a projective representation of $\mathrm{Sp}(V \oplus V^*)$ on $\mathcal{S}(V)$, i.e. a map $\tau : \mathrm{Sp}(V \oplus V^*) \to \mathrm{GL}(\mathcal{S}(V))$ such that $\tau(gh) = \lambda_{g,h}\tau(g)\tau(h)$. It is not possible to coordinate the scalars in order to obtain an honest representation of $\mathrm{Sp}(V \oplus V^*)$, but it is possible to obtain a representation of a double cover $\widetilde{\mathrm{Sp}}(V \oplus V^*)$, called the metaplectic group. This was shown by A. Weil.

12. Derived categories

Let $\mathcal{A}$ be an abelian category, and $\mathcal{C}(\mathcal{A})$ the category of complexes over $\mathcal{A}$. In our convention, differentials of complexes raise the indices, *i.e.* $d^i : C^i \to C^{i+1}$.

Definition 12.1. Let $\varphi : C \to D$ be a morphism in $\mathcal{C}(\mathcal{A})$. We say that φ is homotopic to zero if there exists a collection of maps $\lambda^k : C^{k+1} \to D^k$ such that

$$\varphi^k = \lambda^k \circ d_C^k + d_D^{k-1} \circ \lambda^{k-1}.$$

We say that two morphisms of complexes are homotopic if the difference is homotopic to zero. Define the homotopy category of $\mathcal{A}$ (denoted $\mathcal{K}(\mathcal{A})$) to have complexes as objects and morphisms given by

$$\mathrm{Hom}_{\mathcal{K}}(\mathcal{A})(C, D) := \text{homotopy equivalence classes in } \mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(C, D).$$

We say that two complexes are homotopy equivalent if they are isomorphic in $\mathcal{K}(\mathcal{A})$.

Exercise 12.2.
- (i) The class of morphisms homotopic to zero is closed under both left and right compositions with arbitrary morphisms.
- (ii) Homotopic morphisms induce the same morphisms on cohomologies.

The category $\mathcal{K}(\mathcal{A})$ is additive but not abelian.

Definition 12.3. A morphism $\varphi : C \to D$ in $\mathcal{C}(\mathcal{A})$ (or in $\mathcal{K}(\mathcal{A})$) is called a quasi-isomorphism if the cohomologies $H^k(\varphi)$ are isomorphisms for any k.

The derived category will be defined as the localization of $\mathcal{K}(\mathcal{A})$ by quasi-isomorphisms. The idea is that this category includes slightly more information the the cohomologies of the complexes. We will also define derived functors between derived categories, and they will carry more information than the usual derived functors. In particular, we will be able to compose them, and in this way derive the composition of a left exact functor and a right exact functor.

In order to show that the derived categories are well defined we will show that the quasi-isomorphisms satisfy the Ore condition. For this we will need the cone construction.

Definition 12.4. For $(C, d) \in \mathcal{C}(\mathcal{A})$ define $(\mathrm{Cone}(C), \mathrm{Cone}(d)) \in \mathcal{C}(\mathcal{A})$ by

$$\mathrm{Cone}(C)^i := C^i \oplus C^{i+1}, \quad \mathrm{Cone}(d)(a, b) := (da + b, -db).$$

Notation 12.5. For $(C, d) \in \mathcal{C}(\mathcal{A})$ and $k \in \mathbb{Z}$, denote by $C[k]$ the complex given by

$$C[k]^i = C[k + i], \quad d[k]^i = (-1)^k d^{k+i}.$$

Lemma 12.6. *Exercise*

(1) $\mathrm{Cone}(C)$ *is homotopy equivalent to zero.*
(2) $\varphi : C \to D$ *is homotopic to zero if and only if it can be extended to a morphism* $\varphi' : \mathrm{Cone}(C) \to D$.

Lemma 12.7. *Any morphism of complexes is homotopy equivalent both to an epimorphism and to a monomorphism.*

Proof. Since cones are homotopy equivalent to zero, any $\varphi : C \to D$ is homotopy equivalent to the monomorphism $\varphi' : C \to \mathrm{Cone}(C) \oplus D$ given by $\varphi'_k(a) := (a, 0, \varphi_k(a))$ and to the epimorphism $\varphi'' : C \oplus \mathrm{Cone}(D)[-1] \to D$ given by $\varphi''_k(a, b, c) := \varphi_k(a) + c$. $\square$

Let us give some geometric intuition on cones. For every topological space X one can define a contractible space that includes it by $\mathrm{Cone}(X) := X \times [0, 1]/(X \times \{1\})$. Moreover, for any continuous map $\nu : X \to Y$ we can define $\mathrm{Cone}(\nu)$ to be the quotient of $(X \times [0, 1]) \coprod Y$ by the equivalence relation $(x, 0) \sim \nu(x)$. Then $\mathrm{Cone}(\nu)$ includes Y and the quotient is the suspension $S(X) = X \times [0, 1]/(X \times \{0\} \cup X \times \{1\})$. By this analogy we will now define the cone of a morphism.

Definition 12.8. Let $\varphi : C \to D$ be a morphism in $\mathcal{C}(\mathcal{A})$. Define $\mathrm{Cone}(\varphi) \in \mathcal{C}(\mathcal{A})$ by

$$\mathrm{Cone}(\varphi) := (\mathrm{Cone}(C) \oplus D)/\Delta_C,$$

where $(\Delta_C)_i = \{(c, 0, \varphi(c)) \mid c \in C_i\}$. In other words:

$$\mathrm{Cone}(\varphi)_i := D_i \oplus C_{i+1} \text{ with differential given by } d(a, b) = (da + \varphi(b), -db).$$

Lemma 12.9 (Exercise). *(1) The following short sequence of complexes is exact*

$$0 \to D \to \mathrm{Cone}(\varphi) \to C[1] \to 0.$$

Moreover, the connecting morphism in the corresponding long exact sequence of cohomologies is $\mathrm{H}^{i+1}(\varphi)$.

(2) $\mathrm{Cone}(D \to \mathrm{Cone}(\varphi))$ *is homotopy equivalent to* $C[1]$.
(3) $\mathrm{Cone}(\mathrm{Cone}(\varphi) \to C[1])$ *is homotopy equivalent to* $D[1]$.

The triple $C, D, \mathrm{Cone}(\varphi)$ is called a distinguished triangle (or an exact triangle).

The Lemma 12.9 shows that the exact triangles are symmetric (up to shifts), unlike short exact sequences.

Corollary 12.10. *φ is a quasi-isomorphism if and only if $\mathrm{Cone}(\varphi)$ is an acyclic complex.*

Proposition 12.11. *The system of quasi-isomorphisms in $\mathcal{K}(\mathcal{A})$ satisfies the Ore conditions. In other words for any quasi-isomorphism $\mu : C \to D$ and any morphism $q : E \to D$ there exists a quasi-isomorphism $\nu : L \to E$ and a morphism $p : L \to C$ with $\mu \circ p = q \circ \nu$.*

Proof. By Lemma 12.7 we can assume that $\mu \oplus q : C \oplus E \to D$ is an epimorphism. Let $L := \mathrm{Ker}(\mu \oplus q)$, and let $\nu : L \to E$ and $p : L \to C$ be the projections. From the short exact sequence $0 \to L \to C \oplus E \to D \to 0$ we obtain the long exact sequence

$$\cdots \to \mathrm{H}^{i-1}(D) \to \mathrm{H}^i(L) \to \mathrm{H}^i(C) \oplus \mathrm{H}^i(E) \to \mathrm{H}^i(D) \to \mathrm{H}^{i+1}(L) \to \dots$$

Since μ is a quasi-isomorphism, $\mathrm{H}^i(C)$ is mapped isomorphically to $\mathrm{H}^i(D)$, which implies that the morphism $\mathrm{H}^i(L) \to \mathrm{H}^i(E)$ is onto. Since $\mathrm{H}^{i-1}(C)$ is mapped isomorphically to $\mathrm{H}^{i-1}(D)$ we obtain that the map $\mathrm{H}^{i-1}(D) \to \mathrm{H}^i(L)$ is zero and thus the morphism $\mathrm{H}^i(L) \to \mathrm{H}^i(E)$ is an isomorphism. Thus ν is a quasi-isomorphism. □

Definition 12.12. Let $C, D \in \mathcal{K}(A)$. A (C, D)-triple is a triple (E, ν, φ), where $\nu : E \to C$ is a quasi-isomorphism and $\varphi : E \to D$ is a morphism.

We say that two (C, D)-triples (E, ν, φ) and (E', ν', φ') are linked if there exists an (E, E')-triple (L, α, β) such that both α and β are quasi-isomorphisms and

$$\nu \circ \alpha = \nu' \circ \beta, \quad \varphi \circ \alpha = \beta \circ \varphi'.$$

For $L \in \mathcal{K}(A)$, a join of a (C, D)-triple (E, ν, φ) and a (D, L)-triple (M, μ, ψ) is defined to be the (C, L)-triple $(N, \nu \circ \alpha, \psi \circ \beta)$, where (N, α, β) is an (E, M)-triple satisfying $\varphi \circ \alpha = \mu \circ \beta$. Note that the triple (N, α, β) satisfying the condition always exists by Proposition 12.11.

Lemma 12.13. *The link relation is an equivalence relation, and the equivalence class of the join of two equivalence classes of triples is well-defined, i.e. does not depend on the representatives and on the choice of the triple (N, α, β).*

This lemma follows from Proposition 12.11. We leave the deduction as a long exercise.

Definition 12.14. The derived category $D(\mathcal{A})$ is defined by $Ob(D(\mathcal{A})) = Ob(\mathcal{K}(\mathcal{A}))$ and for $C, D \in Ob(D(\mathcal{A}))$,

$$\mathrm{Hom}_{D(\mathcal{A})}(C, D) = \{\text{equivalence classes of } (C, D) - \text{triples}\}.$$

Lemma 12.15 (Exercise)**.** *The derived category $D(\mathcal{A})$ is additive.*

Hint. Let us explain how to add morphisms. Let $C, D \in Ob(D(\mathcal{A}))$, and let $\eta = (E, \nu, \varphi)$ and $\zeta = (L, \mu, \psi)$ be (C, D)-triples. Proposition 12.11 implies that there exists an (E, L)-triple (M, α, β) such that both α and β are quasi-isomorphisms and $\nu \circ \alpha = \mu \circ \beta$. Then η is equivalent to $(M, \nu \circ \alpha, \varphi \circ \alpha)$ and ζ to $(M, \mu \circ \beta, \psi \circ \beta)$. We define their sum to be (the equivalence class of) $(M, \nu \circ \alpha, \varphi \circ \alpha + \psi \circ \beta)$. ☐

Note that the derived category is not abelian. Rather, it is a triangulated category. Note that we have well-defined cohomology functors $H^i : D(\mathcal{A}) \to \mathcal{A}$.

Definition 12.16. The truncation functors are defined as

$$\tau^{\leq n}(X) := \left(\dots \to X^{n-1} \to \ker\left(X^n \to X^{n+1}\right) \to 0 \to \dots\right)$$
$$\tau^{\geq n}(X) := \left(\dots \to 0 \to \operatorname{coker}\left(X^{n-1} \to X^n\right) \to X^{n+1} \to \dots\right)$$

Then we have natural transformations $\tau^{\leq n}(X) \to X$, $X \to \tau^{\geq n}(X)$, which are isomorphisms if $\mathrm{H}^k(X) = 0$ for any $k > n$ (resp. $k < n$).

$\tau^{\geq n}$ (resp. $\tau^{\leq n}$) is a (co)reflection onto the subcategories of complexes bounded from above (below). For any X the morphisms $\tau^{\leq n}X \to X \to \tau^{\geq n+1}X$ form an exact triangle.

Definition 12.17. For a subset $S \subset \mathbb{Z}$ define $D^S(\mathcal{A})$ to be the subcategory of $\mathcal{D}(\mathcal{A})$ consisting of objects C with $H^k(C) = 0$ for $k \notin S$. Define $D^b(\mathcal{A}) := \bigcup_{\text{finite} S} D^S(\mathcal{A})$.

Remark 12.18. $D^b(\mathcal{A})$ is equivalent to the category of bounded complexes, with link relation through bounded complexes. We will not have time to prove that.

Lemma 12.19. $\mathcal{A} \cong D^{\{0\}}(\mathcal{A})$

Proof. The functors are given by $A \mapsto (\dots \to 0 \to A \to 0 \to \dots)$ and $C \mapsto \mathrm{H}^0(C)$. One composition is the identity. To see that the other composition is isomorphic to identity consider the isomorphisms $C \to \tau^{\geq 0}C$ and $\mathrm{H}^0(C) \to \tau^{\geq 0}C$. ☐

We will say that an object is glued from two others if together they form an exact triangle. We will say that it is glued from some set S of objects if either it lies in S or it is glued from two others, each of which is glued from S.

Exercise 12.20. Let $a \leq b \in \mathbb{Z}$ and let $I := \mathbb{Z} \cap [a, b]$. Then any $D^I(\mathcal{A})$ is glued from $D^{\{a\}}(\mathcal{A}), D^{\{a+1\}}(\mathcal{A}), \dots D^{\{b\}}(\mathcal{A})$.

Definition 12.21. A bicomplex in $\mathcal{A}$ is a collection of objects $B^{ij} \in \mathcal{A}$ parameterized by $\mathbb{Z}^2$ and two collections of morphisms $d_1^{ij} : B^{ij} \to B^{i+1,j}$ and $d_2^{ij} : B^{ij} \to B^{i,j+1}$ such that $d_1^2 = 0$, $d_2^2 = 0$, and $d_1 d_2 + d_2 d_1 = 0$.

For a bicomplex $B = (B^{ij}, d_1^{ij}, d_2^{ij})$ define its total complex $(Tot(B), d)$ by

$$(Tot(B))^k := \bigoplus_{i+j \geq k} B^{ij}, \quad d = d_1 + d_2.$$

Note that we can obtain a bicomplex from a complex of complexes by changing the sign of differentials in every odd column.

Lemma 12.22 (Grothendieck)**.** *Let (B, d_1, d_2) be a bicomplex, and assume that d_1 is acyclic, and on any diagonal $i + j = k$, $B^{ij} = 0, i \ll 0$. Then its total complex* $\operatorname{Tot} B$ *is acyclic.*

Proof. Let $c \in Tot(B)^k$ with $dc = 0$. Let N be s.t. $B^{i,k-i} = 0$ for all $i > N$.

We want to show that $c = dx$ for some $x \in Tot(B)^{k-1}$. We do this by induction on l s.t. $c^{i,k-i} = 0$ for all $i < N + 1 - l$. As a base we take $l = 0$. Then $c = 0$. For the induction step, assume $c^{i,k-i} = 0$ for all $i < N + 1 - l$, and let $\alpha := c^{N+1-l,k-N-1+l}$. Then $d_1\alpha = 0$, thus $\alpha = d_1\beta$ for some $\beta \in B^{N+1-l,k-N+l}$. Then $c \sim c' := c - d\beta$, and $c'^{N+1-l,k-N-1+l} = 0$. Thus $c' = dx'$ by the induction hypothesis. Now, $c = d(\beta + x')$. □

Corollary 12.23. *If $\nu : B \to B'$ is an isomorphism of bicomplexes that satisfy the support condition as above. Suppose ν is a d_1-quasi-isomorphism. Then* $\operatorname{Tot}\nu$ *is a quasi-isomorphism.*

Corollary 12.24. *If B is acyclic except at row 0 and satisfies the support condition as above then* $\operatorname{Tot} B$ *is quasi-isomorphic to the cohomology complex $H^{0,\bullet}(B)$.*

Proof. Let $B^{\bullet j}$ denote the j-th column of B. Consider the exact triangle of complexes:

$$\tau^{<0}B^{\bullet j} \to B^{\bullet j} \to \tau^{\geq 0}B^{\bullet j}.$$

The first one is acyclic, and thus $B^{\bullet j} \to \tau^{\geq 0}B^{\bullet j}$ is a quasi-isomorphism. We get a d_1-quasi-isomorphism of bicomplexes $B \to \tau^{i\geq 0}B$. By the previous corollary this implies a quasi-isomorphism $Tot(B) \to Tot(\tau^{i\geq 0}B)$.

In the same way, the exact triangle

$$\tau^{<1}(\tau^{\geq 0}B^{\bullet j}) \to \tau^{\geq 0}B^{\bullet j} \to \tau^{\geq 1}B^{\bullet j}.$$

gives a quasi-isomorphism $\tau^{i<1}(\tau^{i\geq 0}B) \to \tau^{i\geq 0}B$, and by taking total complexes, a quasi-isomorphism $H^{0,\bullet}(B) \to Tot(\tau^{i\geq 0}B)$. Together, we get isomorphisms in the derived category between $Tot(B)$, $Tot(\tau^{i\geq 0}B)$ and $H^{0,\bullet}(B)$. □

Now we would like to define derived functors. Suppose that $\mathcal{A}$ has enough injective objects.

Lemma 12.25. *Any $C \in \mathcal{C}^{\geq 0}(\mathcal{A})$ has an injective resolution, i.e. is quasi-isomorphic to a complex consisting of injective objects.*

Proof. First of all, let us show that C can be embedded into an injective complex. Embed C^0 into an injective I^0, and C^1 into (an injective) I^1. Then the composed map $C^0 \to I^1$ can be lifted (by the injectivity of I^1) to $d^0 : I^0 \to I^1$. Then we embed C^2 into (an injective) I^2, and lift the map $C^1/d^0_C(C^0) \to I^2$ to a map $(d^2)' : I^1/d^0(I^0) \to I^2$. We continue building I by induction.

Now we embed C into an injective complex I^0, then I^0/C into I^1 and so on. In this way we construct a bicomplex $0 \to I^0 \to I^1 \to \dots$. By Corollary 12.24 the total complex will be quasi-isomorphic to C. □

Lemma 12.26 (Exercise)**.** *Let I, J be bounded on the left complexes consisting of injective objects, and let $\varphi : I \to J$ be a quasi-isomorphism. Then φ is an isomorphism in the homotopic category.*

Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor.

Definition 12.27. For any $C \in D(\mathcal{A})$ choose an injective resolution I and define $DF(C) := F(I)$. This defines a functor $DF : D(\mathcal{A}) \to D(\mathcal{B})$.

We say that an object $X \in \mathcal{A}$ is F-acyclic if $DF(X) \in D^{\{0\}}\mathcal{B}$.

Proposition 12.28 (Exercise)**.** *Let C be a bounded on the left complex consisting of F-acyclic objects. Then $DF(C) \cong F(C)$.*

13. The derived category of D-modules

From now on we assume for simplicity that X is a quasi-projective variety. Let $\mathcal{M}(\mathcal{O}_X)$ denote the category of sheaves of $\mathcal{O}_X$-modules, and $\mathcal{M}_{q.c.}(\mathcal{O}_X) \subset \mathcal{M}(\mathcal{O}_x)$ denote the subcategory of quasi-coherent sheaves of $\mathcal{O}_X$-modules.

Proposition 13.1. *The category $\mathcal{M}_{q.c.}(\mathcal{O}_X)$ has enough objects injectives, that are also injective as object in $\mathcal{M}(\mathcal{O}_X)$.*

Proof. For affine X, we take duals to free objects: $\mathrm{Hom}_{\mathbb{K}}(\mathcal{O}(X)^J, \mathbb{K})$. For non-affine X, we take localizations of $\mathcal{O}(X)$-modules of the form $\bigoplus_i (j_{U_i})_* I_i$, where $\{U_i\}$ is an open affine cover, and I_i are injective quasi-coherent sheaves on U_i. □

Corollary 13.2. *The embedding of $D^b(\mathcal{M}_{q.c.}(\mathcal{O}_X))$ into the category $D^b_{q.c.}(\mathcal{M}(\mathcal{O}_X))$ consisting of bounded complexes with quasi-coherent cohomologies is an equivalence of categories.*

Proof. It is enough to show that every complex C in $D^b_{q.c.}(\mathcal{M}(\mathcal{O}_X))$ is quasi-isomorphic to a complex consisting of quasi-coherent injective sheaves of $\mathcal{O}_X$-modules.

First of all, using the truncation functors we showed before that we can assume that $C_{-i} = 0$ for any $i > 0$. Now, $\mathrm{Ker}(d^0)$ is quasi-coherent, and thus can be embedded into an injective quasi-coherent I^0. Denote this embedding by φ^0. Next, we consider $(I^0 \oplus C^1)/(\varphi^0 \oplus d)(C^0)$ and let φ^1 be an embedding of this sheaf into an injective quasi-coherent I^1. Restricting φ^1 to the first coordinate we obtain a map $d_I^0 : I^0 \to I^1$. Continuing by induction we build a complex I consisting of quasi-coherent injective sheaves, and a map of complexes $\varphi : C \to I$.

Let us show that φ is indeed a quasi-isomorphism. We have

$$H_I^0 = \mathrm{Ker}(d_I^0) = (I^0 \oplus 0) \cap \mathrm{Im}(\varphi^0 \oplus d^0) \cong \mathrm{Ker}(d^0) = H_C^0$$

Similarly, each φ^i defines an isomorphism between $\mathrm{Ker}\, d_C^i$ and $\mathrm{Ker}\, d_I^i$, and between $\mathrm{Im}\, d^i$ and $\mathrm{Im}\, d_D^i$. Thus, φ is a quasi-isomorphism. □

Lemma 13.3 (Exc*)**.** *Let $\mathcal{F} \in \mathcal{M}_{q.c.}(\mathcal{O}_X)$ and let $U \subset X$ be an open subset. Let $\mathcal{H} \subset \mathcal{F}|_U$ be a coherent subsheaf. Then there exists a (non-unique) coherent subsheaf $\mathcal{H}' \subset \mathcal{F}$ such that $\mathcal{H}'|_U = \mathcal{H}$.*

Corollary 13.4. *Any quasi-coherent sheaf is a direct limit of coherent ones.*

Corollary 13.5. *Any coherent sheaf is a quotient of a locally-free coherent sheaf.*

Let $\mathcal{M}_{coh}(\mathcal{O}_X) \subset \mathcal{M}_{q.c.}(\mathcal{O}_X)$ denote the subcategory of coherent sheaves, and let $D^b_{coh}(\mathcal{M}_{q.c.}(\mathcal{O}_X))$ denote the subcategory consisting of complexes with coherent cohomologies.

Corollary 13.6. *The embedding of $D^b(\mathcal{M}_{coh}(\mathcal{O}_X)$ into $D^b_{coh}(\mathcal{M}_{q.c.}(\mathcal{O}_X))$ is an equivalence of categories.*

Proof. Let

$$0 \to C^0 \to C^1 \to \cdots \to C^n \to 0$$

be a bounded complex with coherent cohomologies. Then there exists a coherent free sheaf K^n and an epimorphism $\varphi^n : K^n \twoheadrightarrow C^n/d^{n-1}(C^{n-1})$. Let $C^{n-1} \times_{C^n} K := \mathrm{Ker}(d^{n-1} - \varphi^n : C^{n-1} \oplus K \to C^n)$. Then there exists a coherent free sheaf K^{n-1} and an epimorphism $\varphi^{n-1} : K^{n-1} \twoheadrightarrow C^{n-1} \times_{C^n} K$. Continuing in this way we build a complex of free coherent sheaves and a quasiisomorphism from it to C. $\square$

Now let $D^b(\mathcal{D}_X)$ denote the bounded derived category of left $\mathcal{D}_X$-modules. Recall that the category of $\mathcal{D}_X$-modules has enough injectives and enough locally projectives.

Now we would like to construct the pullback and pushforward functors between derived categories of $\mathcal{D}$-modules. We start with pullback.

13.1. **The pullback $\pi^!$.** Recall that for affine varieties we have the pullback functor $\pi^0(\mathcal{H}) = \mathcal{H} \otimes_{\mathcal{O}(Y)} \mathcal{O}(X)$, on which we defined the action of $\mathcal{D}(X)$. For general maps of general varieties, let $\pi^\bullet$ denote the pullback of sheaves and define

$$\pi^0(\mathcal{H}) := \pi^\bullet(\mathcal{H}) \otimes_{\pi^\bullet \mathcal{O}_Y} \mathcal{O}_X$$

This functor is right-exact since $\pi^\bullet$ is exact, and tensor product is right-exact. Define $\pi^!$ to be the derived functor of π^0, shifted by the difference of dimensions:

$$\pi^! := L\pi^0[\dim X - \dim Y]$$

Since $(\pi \circ \tau)^0 \cong \tau^0 \circ \pi^0$, we have

Lemma 13.7. $(\pi \circ \tau)^! \cong \tau^! \circ \pi^!$

This allows to analyze the pullback functor by decomposing the map π into simple parts. Any map can be decomposed as a composition of open embedding, closed embedding, and projective projection.

For an open embedding $j : U \hookrightarrow Y$, $j^!(\mathcal{F}) = \mathcal{F}|_U$. For a projection $Y \times \mathbb{P}^n \twoheadrightarrow Y$,

$$\pi^! \mathcal{F} = \mathcal{F} \boxtimes \mathcal{O}_{\mathbb{P}^n}[n]$$

Let us now analyze $i^!$ for a closed embedding $i : Z \hookrightarrow Y$. For this purpose we need to analyze $i^! \mathcal{D}_Y$. This analysis will be based on the following lemma. Let ω_Y denote the invertible sheaf of top differential forms on Y.

Lemma 13.8. $\mathrm{Ext}^j_{i_\bullet \mathcal{O}_Y}(\mathcal{O}_Z, \omega_Y) \cong \begin{cases} \omega_Z & j = \dim Y - \dim Z \\ 0 & \textit{otherwise} \end{cases}$

Proof. Step 1 Enough to prove this locally, since the isomorphism is canonical and thus glues on intersections.

Step 2 Can assume that Y is affine, and that Z is given by a single equation t.

Step 3 In this case, we have a free resolution of $\mathcal{O}(Z)$: $0 \to (t) \to \mathcal{O}(Y) \to 0$. Taking Hom into ω_Y we get $0 \to \omega_Y \xrightarrow{t\cdot} \omega_Y$, and the cohomologies are 0 and ω_Z.

$\square$

Now recall the functor $i^+ : \mathcal{M}^r(\mathcal{D}_Y) \to \mathcal{M}^r(\mathcal{D}_Z)$ of taking sections strongly supported at Z.

$$i^+(\mathcal{H}) := \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}_{Z\to Y}, \mathcal{H}) = \mathcal{H}om_{\mathcal{D}_Y}(i^\bullet(\mathcal{D}_Y) \otimes_{i^\bullet(\mathcal{O}_Y)} \mathcal{O}_Z, \mathcal{H}) \cong \mathcal{H}om_{\mathcal{O}_Y}(i_\bullet \mathcal{O}_Z, \mathcal{H})$$

Recall also the equivalence of categories E between left and right $\mathcal{D}$-modules given by tensoring over $\mathcal{O}$ with ω. From the previous lemma we get

Corollary 13.9. $Li^0(\mathcal{D}_Y)[\dim Z - \dim Y] \cong Ri^+(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \omega_Y)$

This implies

Theorem 13.10. $i^! := Li^0[\dim Z - \dim Y] \cong Ri^+ \circ E.$

Proof. The previous corollary gives a canonical isomorphism for free modules, and since it's canonical it extends to locally free modules, and thus to complexes of locally free modules. Now, any complex is quasi-isomorphic to a complex of locally free modules. □

This functor does not in general preserve coherence. This can be seen for $i : pt \to Y$, $i^! \mathcal{D}_Y$ is infinite-dimensional. We will show later that a complex C has holonomic cohomologies if and only if $i^!_x C$ has finite-dimensional cohomologies for every point $x \in Y$. But for smooth maps π, $\pi^!$ does preserve coherence - that is, being (locally) finitely-generated as $\mathcal{D}$-modules. Smooth maps are maps such that their differentials are onto. In differential geometry this is called submersion.

Theorem 13.11. *If π is smooth then $\pi^!$ preserves coherence.*

Sketch of proof. A smooth map can be decomposed as the composition of a projection and an etale map (i.e. a map with differentials being isomorphisms. For a projection $p : X \times Y \to Y$, $p^!(\mathcal{H}) = \mathcal{O}_X \boxtimes \mathcal{H}$, which clearly preserves coherence. Pullbacks under etale maps preserve coherence since as a complex of $\mathcal{O}$-modules this is the derived pullback of $\mathcal{O}$-modules and this operation preserves $\mathcal{O}$-generation, and since $d_x\pi$ is an isomorphism for every point. □

Definition 13.12. For two complexes of left modules $\mathcal{F}, \mathcal{H} \in D^b(\mathcal{M}(\mathcal{D}_X))$, define

$$\mathcal{F} \otimes^! \mathcal{H} := \Delta^!(\mathcal{F} \boxtimes \mathcal{H}),$$

where $\Delta : X \hookrightarrow X \times X$ is the diagonal embedding.

13.2. **The pushforward π_*.** Recall that for affine X and Y, π_0 is defined by $\pi_0(M) = M \otimes_{\mathcal{D}_X} \mathcal{D}_Y$. For general varieties, we will need also to compose this functor with the pushforward of sheaves, defined by $\pi_\bullet(\mathcal{F})(U) = \mathcal{F}(\pi^{-1}(U))$. There is a difficulty here: tensor product is a right-exact functor, while $\pi_\bullet$ is left exact, and the composition will not be exact on either side. However, in the derived category we can compose their derived functors. We define

$$\pi_*(\mathcal{F}) := R\pi_\bullet(\mathcal{F} \otimes^L_{\mathcal{D}_X} \mathcal{D}_{X\to Y})$$

To compute this, we can use locally projective resolutions for the tensor product, and injective resolutions for $\pi_\bullet$. Another way to compute is again to decompose the map as a composition of a closed embedding, an open embedding and a projective projection.

For a closed embedding i, the i_0 is an equivalence with a subcategory of $\mathcal{M}^r(\mathcal{D}_Y)$, and i_* is given by just applying i_0 to all the sheaves in a complex.

For an open embedding $j : U \hookrightarrow Y$, we have $\mathcal{D}_{U\to Y}|_U = \mathcal{D}_U$ and thus $j_* = Rj_\bullet$.

Proposition 13.13. *Let $U \subset Y$ be an open subset, and let Z be its complement. Let $j : U \hookrightarrow Y$ and $i : Z \hookrightarrow Y$ denote the embeddings. Let $\mathcal{F} \in D^b(\mathcal{M}(\mathcal{D}_Y))$, and let $\mathcal{F}_Z := i_*i^!\mathcal{F}$ and $\mathcal{F}_U := j_*j^!\mathcal{F} = j_*(\mathcal{F}|_U)$. Then we have a distinguished triangle*

$$\mathcal{F}_Z \to \mathcal{F} \to \mathcal{F}_U.$$

Proof. Step 1 $\mathcal{F}_Z = R\Gamma_Z\mathcal{F}$ by Kashiwara's lemma.

Step 2 For any sheaf $\mathcal{G}$, we have

$$0 \to \Gamma_Z\mathcal{G} \to \mathcal{G} \to j_\bullet(\mathcal{G}|_U),$$

and if $\mathcal{G}$ is injective then the rightmost map is an epimorphism.

Step 3 We can replace $\mathcal{F}$ by its injective resolution, i.e. by a complex of injective sheaves of modules quasi-isomorphic to $\mathcal{F}$.

□

It is left to compute p_* for the projection $p : \mathbb{P}^n \to pt$. Since the category of (sheaves of) D-modules on $\mathbb{P}^n$ is generated by $\mathcal{D}_{\mathbb{P}^n}$, it is enough to compute $p_*(\mathcal{D}_{\mathbb{P}^n})$.

Lemma 13.14. *For $p : X \to pt$, $\mathsf{H}^i(p_*(\mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X)) \cong \mathsf{H}^i(X, \omega_X)$.*

Proof. $\mathcal{D}_{X\to pt} = \mathcal{O}_X$. Thus

$$\mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to pt} \cong \omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_X \cong \omega_X$$

Also, $p_\bullet = \Gamma$, thus $Rp_\bullet = R\Gamma$, and has cohomologies $\mathsf{H}^i(X, \cdot)$. □

We will use without proof the following well-known lemma in algebraic geometry.

Lemma 13.15. $\mathsf{H}^i(\mathbb{P}^n, \omega_{\mathbb{P}^n}) \cong \begin{cases} \mathbb{K} & i = n \\ 0 & i \neq n \end{cases}$

Corollary 13.16. *For $p : \mathbb{P}^n \to pt$, $p_*(\mathcal{D}_{\mathbb{P}^n} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \omega_{\mathbb{P}^n}) \cong \mathbb{K}[-n]$.*

Proof. By the previous lemma and corollary, $p_*(\mathcal{D}_{\mathbb{P}^n} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \omega_{\mathbb{P}^n})$ is acyclic away of index n, and the cohomology there is $\mathbb{K}$. Thus it is $\mathbb{K}[-n]$. □

Corollary 13.17. *For projective morphisms π, π_* preserves coherence.*

Proof. A projective morphism is a composition of a closed embedding with a projective projection. For a closed embedding this is clear. For a projective projection $\pi : Y \times \mathbb{P}^n \to Y$, we have $\pi_*(\mathcal{F} \boxtimes \mathcal{G}) = \mathcal{F} \boxtimes p_*\mathcal{G}$, where $p : \mathbb{P}^n \to pt$. Since p_* preserves coherence by the previous corollary, and exterior products of the form $\mathcal{F} \boxtimes \mathcal{G}$ generate the category of D-modules on Cartesian product, π_* preserves coherence. □

Let us now compute the direct image of ω_X. For this we will need a locally free resolution. We will use the algebraic de-Rham complex DR_X

$$0 \to \mathcal{O}_X \to \Omega^1_X \to \cdots \to \Omega^n_X \to 0, \tag{4}$$

where $n := \dim X$. The Ω^i_X are locally projective $\mathcal{O}_X$-modules, but the differential in this complex is not a morphism of $\mathcal{O}_X$-modules. It is the classic exterior derivative of differential forms. Here are two ways to define it:

(1) In local coordinates: $d\omega = \sum_i \frac{\partial \omega}{\partial x_i} \wedge dx_i$
(2) Axiomatically:
   (a) For $f \in \mathcal{O}_X$, $df(v) = vf$ for any vector field v
   (b) $d(d(w)) = 0$ (enough to require this for any 0-form $\omega \in \mathcal{O}(X)$)
   (c) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$

The cohomologies of the algebraic de-Rham complex are complicated.

Generalizing, we can construct the de-Rham complex of $DR(M)$ for any left $\mathcal{D}_X$-module M:

$$0 \to M \to \Omega^1_X \otimes_{\mathcal{O}_X} M \to \cdots \to \Omega^n_X \otimes_{\mathcal{O}_X} M \to 0$$

Indeed, $\Omega^i_X \otimes_{\mathcal{O}_X} M$ is the sheaf of M-valued forms, and in the axiomatic definition of differential, we only needed to know what vf is for any vector field v, so we replace axiom $f \in \mathcal{O}_X$ by $m \in M$ in axiom (a). As before, the elements of the complex are $\mathcal{O}_X$-modules, and the differentials are merely linear maps. But if we take $M := \mathcal{D}_X$ then the complex (and its differentials) becomes a complex of right $\mathcal{D}_X$ modules:

$$0 \to \mathcal{D}_X \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \cdots \to \Omega^n_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \to 0 \tag{5}$$

Lemma 13.18. (5) *is a locally projective resolution of* $\omega_X = \Omega^n_X$.

To prove this lemma we will need the dual one. Let τ_X denote the tangent sheaf of X, and for any $0 \le i \le \dim X$, let $\Lambda^i(\tau_X)$ be its exterior power (over $\mathcal{O}_X$). Define $d : \mathcal{D}_X \otimes_{\mathcal{O}_X} \Lambda^i(\tau_X) \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \Lambda^{i-1}(\tau_X)$ by

$$d(P \otimes \theta_1 \wedge \cdots \wedge \theta_k) := \sum_i (-1)^{i+1} P\theta_i \otimes \theta_1 \wedge \cdots \wedge \hat{\theta}_i \wedge \cdots \wedge \theta_k +$$
$$+ \sum_{i<j} (-1)^{i+j} P \otimes [\theta_i, \theta_j] \wedge \theta_1 \wedge \cdots \wedge \hat{\theta}_i \wedge \cdots \wedge \hat{\theta}_j \cdots \wedge \theta_k$$

This defines a complex

$$0 \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \Lambda^n(\tau_X) \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \Lambda^{n-1}(\tau_X) \to \cdots \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \tau_X \to \mathcal{D}_X \to \mathcal{O}_X \to 0 \tag{6}$$

Lemma 13.19. *The complex* (6) *is acyclic, and thus defines a locally free resolution of* $\mathcal{O}_X$.

Proof. It is enough to show that the associated graded complex with respect to the geometric filtration is acyclic. This associated graded is the pushforward under the affine map $\pi : T^*X \to X$ of the complex

$$\mathcal{O}_{T^*X} \otimes_{\pi^\bullet \mathcal{O}_X} \Lambda^n \pi^\bullet \tau_X \to \mathcal{O}_{T^*X} \otimes_{\pi^\bullet \mathcal{O}_X} \Lambda^{n-1} \pi^\bullet \tau_X \to \dots \mathcal{O}_{T*X} \to i_\bullet \mathcal{O}_X,$$

where $i : X \to T^*X$ is the zero section. The latter complex is acyclic because it is the Kozhul complex of $i_\bullet \mathcal{O}_X$ (Exc - verify this). Since π is an affine map, the pushforward to X is also acyclic. $\square$

Now, Lemma 13.18 follows by tensoring with $\omega_X \cong \Omega^n_X$ over $\mathcal{O}_X$.

Corollary 13.20. *For* $\pi : X \to pt$, $\pi_*(\omega_X) = R\Gamma(DR_X)$

Proof. By definition, $\pi_*(\omega_X) = R\Gamma(D_{X \to pt} \otimes^L_{\mathcal{D}_X} \omega_X)$. Also by definition we have $\mathcal{D}_{X\to pt} = \mathcal{O}_X$. To compute $\otimes^L_{\mathcal{D}_X}$ we take a locally projective resolution (5) of ω_X. Tensoring it over $\mathcal{D}_X$ with $\mathcal{O}_X$ we obtain DR_X. $\square$

Remark 13.21. More generally, for every submersion $\pi : X \to Y$ one can define the relative de-Rham complex $DR_{X/Y}$. The same argument will show that $\pi_*\omega_X$ in this case is $R\pi_\bullet(DR_{X/Y})$, as a complex of $\mathcal{O}_Y$-modules. How to recover the $\mathcal{D}_Y$-module structure? In general it is hard to say, but if we assume further that π is proper then one can recover the $\mathcal{D}_Y$-module structure on the cohomologies of this complex. The cohomologies will be $\mathcal{O}_Y$-coherent, and thus will be just vector bundles with flat connections. The additional information is the connections. They are known in the literature as Gauss-Manin connections.

Theorem 13.22. *Let $\pi : Y \to Z$ and $\tau : X \to Y$ be morphisms of algebraic varieties. Then $(\pi \circ \tau)_* \cong \pi_* \circ \tau_*$.*

Proof. Step 1 $\mathcal{D}_{X\to Z} \cong \mathcal{D}_{X\to Y} \otimes_{\tau^\bullet(\mathcal{D}_Y)} \tau^\bullet(\mathcal{D}_{Y\to Z}) \cong \mathcal{D}_{X\to Y} \otimes^L_{\tau^\bullet(\mathcal{D}_Y)} \tau^\bullet(\mathcal{D}_{Y\to Z})$.

Indeed, for the first isomorphism we have

$$\mathcal{D}_{X\to Z} = (\pi\cdot\tau)^\bullet\mathcal{D}_Z \otimes_{(\pi\cdot\tau)^\bullet\mathcal{O}_Z} \mathcal{O}_X \cong \tau^\bullet(\pi^\bullet\mathcal{D}_Z \otimes_{\pi^\bullet\mathcal{O}_Z} \mathcal{O}_Y) \otimes_{\tau^\bullet\mathcal{D}_Y} \pi^\bullet\mathcal{D}_Y \otimes_{\pi^\bullet\mathcal{O}_Y} \mathcal{O}_X$$

For the second isomorphism, we note that $\mathcal{D}_Z$ is a locally free $\mathcal{O}_Z$- module, and thus $\mathcal{D}_{Y\to Z}$ is a locally free $\mathcal{O}_Y$-module. Thus $\tau^\bullet(\mathcal{D}_{Y\to Z})$ is "flat with respect to $\mathcal{D}_{X\to Y}$".

Step 2 For any $\mathcal{F} \in D^b(\mathcal{M}^r(\mathcal{D}_Y))$, and $\mathcal{G} \in D^b(\mathcal{M}(\tau^\bullet\mathcal{D}_Y))$ we have

$$\mathcal{F} \otimes^L_{\mathcal{D}_Y} R\tau_\bullet(\mathcal{G}) \cong R\tau_\bullet(\tau^\bullet\mathcal{F} \otimes^L_{\tau^\bullet\mathcal{D}_Y} \mathcal{G}))$$

Since the question is local on Y, we can assume that that Y is affine and that $\mathcal{F}$ is free (replacing it by a locally free resolution). Now the statement follows since $R\tau_\bullet$ commutes with direct sums.

Step 3 Let $M \in D^b(\mathcal{M}^r(\mathcal{D}_X))$. Then

$$(\pi\circ\tau)_*(M) = R\pi_\bullet R\tau_\bullet(M \otimes^L_{\mathcal{D}_X} \mathcal{D}_{X\to Z}) \cong R\pi_\bullet R\tau_\bullet(M \otimes^L_{\mathcal{D}_X} \mathcal{D}_{X\to Y} \otimes^L_{\tau^\bullet\mathcal{D}_Y} \tau^\bullet(\mathcal{D}_{Y\to Z})) \cong$$
$$R\pi_\bullet(R\tau_\bullet(M \otimes^L_{\mathcal{D}_X} \mathcal{D}_{X\to Y}) \otimes^L_{\mathcal{D}_Y} \mathcal{D}_{Y\to Z}) \cong \pi_*\tau_* M$$

□

13.3. **Functors between modules on singular varieties.** We mostly assume our varieties to be smooth. In particular, this allows to identify the categories of left and right D-modules, using $\cdot \otimes_{\mathcal{O}_X} \omega_X$. Thus we can also speak of pullback of right D-modules.

Over a singular variety Z, we define the derived category of complexes of right D-modules in the same way as we did for affine varieties - by embedding Z into a smooth variety X, and define this category to be subcategory of $D^b(\mathcal{M}^r(\mathcal{D}_X))$ consisting of complexes with cohomologies supported in Z. This is justified by Kashiwara's theorem, and as before the same theorem shows that this does not depend on the embedding. Note that such an embedding exists, since we consider quasi-projective varieties, that by definition are closed subvarieties of open subsets of a projective space. Otherwise we would have to embed locally, which also works but is more messy.

How to define the functors? We will need the following elementary exercise.

Exercise 13.23. Let X, Y be smooth, and let $\pi : X \to Y$ be a morphism. Let $\mathcal{F} \in D^b(\mathcal{M}^r(\mathcal{D}_X))$ and $\mathcal{H} \in D^b(\mathcal{M}(\mathcal{D}_Y))$. Then $\operatorname{Supp} \pi_*\mathcal{F} \subset \overline{\pi(\operatorname{Supp}\mathcal{F})}$ and $\operatorname{Supp} \pi^!\mathcal{H} \subset \pi^{-1}(\operatorname{Supp}\mathcal{H})$.

Here, we define the support of a complex to be the union of the supports of the cohomologies.

Suppose we have a map $W \to Z$ between singular varieties. First of all, we can embed Z into a smooth Y, and consider now the map $\pi : W \to Y$. By the previous exercise on the preservation of supports, it will be enough to define the pullback and the pushforward for this map. Embed W into a smooth X, and then embed W into $X \times Y$ using the graph of π. This is a closed embedding, and thus we can identify the category of D-modules on $X \times Y$ with the category of (bounded complexes of) D-modules supported on W. Now, from $X \times Y$ we can easily push to Y, and vice versa - pull from Y to $X \times Y$. This pullback will be just exterior product with $\omega_X[\dim X]$ (ω_X and not $\mathcal{O}_X$ since we consider right D-modules in this subsection, since that is what singular varieties need). Then, to define pullback to W we will need to apply the derived functor of the functor of taking sections supported at W.

14. Base change, and adjunction

14.1. **Base change.** Let $\pi : X \to Y$ and $\tau : Z \to Y$ be morphisms of algebraic varieties, and let $W := X \times_Y Z$ be the fiber product. Let $p : W \to Z$ and $t : W \to X$ be the natural projections.

$$\begin{tikzcd} W \arrow[r, "t"] \arrow[d, "p"] & X \arrow[d, "\pi"] \\ Z \arrow[r, "\tau"] & Y \end{tikzcd}$$

Theorem 14.1 (Base change)**.** $p_* t^! \cong \tau^! \pi_*$

Proof. Case 1 $Z = Y \times Q$, and $Z \to Y$ is the projection. Then the statement is that direct image commutes with $\boxtimes$.

Case 2 τ is an open embedding. Then the claim is that the direct image is a local operation.

Case 3 τ is a closed embedding. Then W is the preimage of Z in X. Let $U := Y \setminus Z$. Then we have the exact triangles

$$\begin{tikzcd} (\pi_*\mathcal{F})_Z \arrow[r] & \pi_*\mathcal{F} \arrow[r] \arrow[d, equal] & (\pi_*\mathcal{F})_U \arrow[d, "\simeq"] \\ \pi_*(\mathcal{F}_W) \arrow[r] & \pi_*\mathcal{F} \arrow[r] & \pi_*(\mathcal{F}_{\pi^{-1}(U)}) \end{tikzcd}$$

From the isomorphisms of the two rightmost terms (compatible with the rows) we get that the leftmost terms are isomorphic as well. The isomorphism is not canonical for exact triangles in general, but it is unique up to maps from the leftmost term to the rightmost. In our case, there are no maps between $(\pi_*\mathcal{F})_Z$ and $(\pi_*\mathcal{F})_U$, and thus the isomorphism $(\pi_*\mathcal{F})_Z \simeq \pi_*(\mathcal{F}_W)$ is unique. $\square$

Let us give an application of the base change to families of D-modules. Intuitively, a family of D-modules parameterized by an algebraic variety Y is the following information: for every point $y \in Y$ a pair consisting of an algebraic variety X_y and a $\mathcal{D}_{X_y}$-module. The pair should depend "algebraically" on the point y, and here is the way to formalize this notion.

Definition 14.2. A family of D-modules parameterized by an algebraic variety Y is a morphism of algebraic varieties $\pi : X \to Y$ and a $\mathcal{D}_X$-module (or rather a complex of modules) $\mathcal{F} \in \mathcal{D}^b(\mathcal{M}(\mathcal{D}_X))$. For any $y \in Y$ we have the pair $X_y := \pi^{-1}(y)$ and $\mathcal{F}_y := i_y^! \mathcal{F}$ where $i_y : X_y \hookrightarrow X$ is the embedding.

For every map $\tau : Z \to Y$ we can pullback the family: $X_z := Y_{\tau(z)}$ and $\mathcal{F}_z := \mathcal{F}_{\tau(z)}$. It is easy to see that the pullback of an algebraic family is an algebraic family. Indeed, let $W := W \times_Y Z$ as before, and let $\mathcal{G} := t^! \mathcal{F}$, where $t : W \to X$ is the natural projection. Then for every $z \in Z$, $X_z \cong W_z$ and $i_z^! \mathcal{G} \cong i_z^! t^! \mathcal{F} \cong i_{\tau(z)}^! \mathcal{F} \cong \mathcal{F}_z$.

Similarly, if we have a map $Z \to X$, we can pullback the family to Z.

Let us now consider a pushforward of an algebraic family. Let $\tau : X \to T$ and $\nu : T \to Y$ be morphisms of algebraic varieties such that $\nu \circ \tau = \pi$. Then we can pushforward $\mathcal{F}$ to T, that is consider $\tau_* \mathcal{F}$. What does this do to fibers? It is natural to assume that $\tau_*(\mathcal{F})_y \cong (\tau_y)_* \mathcal{F}_y$, where $\tau_y : X_y \to T_y$ is the restriction of τ. This indeed follows from the base change theorem: $X \times_T T_y \cong X_y$, and the two ways of going from X to T_y are isomorphic.

Finally, let us show adjointness of pushforward and pullback for projective morphisms.

Theorem 14.3. *For any projective morphism $\pi : X \to Y$, π_* is left adjoint to $\pi^!$, after we identify left modules with right ones. That is for any $\mathcal{F} \in D^b(\mathcal{M}^r(\mathcal{D}_X)$ and any $\mathcal{H} \in D^b(\mathcal{M}(\mathcal{D}_Y))$ we have*

$$\mathrm{Hom}_{\mathcal{D}_Y}(\pi_* \mathcal{F}, \mathcal{H} \otimes_{\mathcal{O}_Y} \omega_Y) \cong \mathrm{Hom}(\mathcal{F}, \pi^! \mathcal{H} \otimes_{\mathcal{O}_X} \omega_X)$$

Proof. Case 1 π is a closed embedding. By Kashiwara's lemma:

$$\mathrm{Hom}_{\mathcal{D}_Y}(\pi_* \mathcal{F}, \mathcal{H} \otimes_{\mathcal{O}_Y} \omega_Y) = \mathrm{Hom}(\pi_* \mathcal{F}, \Gamma_X(\mathcal{H} \otimes_{\mathcal{O}_Y} \omega_Y)) \cong \\ \mathrm{Hom}(\mathcal{F}, \pi^! \Gamma_X(\mathcal{H} \otimes_{\mathcal{O}_Y} \omega_Y)) \cong \mathrm{Hom}(\pi_* \mathcal{F}, \pi^! \mathcal{H} \otimes_{\mathcal{O}_Z} \omega_Z)$$

Case 2 π is a projection $\mathbb{P}^n \times Y \to Y$. Since the question is local on Y, we can assume that Y is affine. Then it is enough to show for $\mathcal{F} \cong \mathcal{D}_{\mathbb{P}^n} \boxtimes \mathcal{D}_Y$ and $\mathcal{H} \cong \mathcal{D}_Y[j]$ for some j. Then

$$\mathrm{Hom}_{\mathcal{D}_Y}(\pi_*(\mathcal{D}_{\mathbb{P}^n} \boxtimes \mathcal{D}_Y), \mathcal{D}_Y[j] \otimes_{\mathcal{O}_Y} \omega_Y) \cong \mathrm{Hom}_{\mathcal{D}_Y}(\mathbb{K}[-n] \boxtimes \mathcal{D}_Y, \mathcal{D}_Y[j] \otimes_{\mathcal{O}_Y} \omega_Y) \cong \\ \mathrm{Hom}_{\mathcal{D}_Y}(\mathcal{D}_Y, \mathcal{D}_Y[j+n] \otimes_{\mathcal{O}_Y} \omega_Y) \cong \mathrm{Hom}_{\mathcal{D}_{\mathbb{P}^n}}(\mathcal{D}_{\mathbb{P}^n}, \omega_{\mathbb{P}^n}[n]) \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathcal{D}_Y}(\mathcal{D}_Y, \mathcal{D}_Y[j+n] \otimes_{\mathcal{O}_Y} \omega_Y) \cong \\ \cong \mathrm{Hom}_{\mathcal{D}_{\mathbb{P}^n \times Y}}(\mathcal{D}_{\mathbb{P}^n} \boxtimes \mathcal{D}_Y, \omega_{\mathbb{P}^n}[n] \boxtimes \ \mathcal{D}_Y[j+n] \otimes_{\mathcal{O}_Y} \omega_Y) \cong \\ \mathrm{Hom}_{\mathcal{D}_{\mathbb{P}^n \times Y}}(\mathcal{D}_{\mathbb{P}^n} \boxtimes \mathcal{D}_Y, \pi^! \mathcal{D}_Y[j] \otimes_{\mathcal{O}_{Y \times \mathbb{P}^n}} \omega_{Y \times \mathbb{P}^n})$$

$\square$

15. Duality

Assume again that X is smooth.

Exercise 15.1. $\mathcal{D}_X \cong (\Delta_X)_*(\omega_X) \otimes_{\mathcal{O}_X} \Omega_X^*$.

Definition 15.2. For every $\mathcal{F} \in D^b_{coh}(\mathcal{M}(\mathcal{D}_X))$, define the dual module by

$$\mathbb{D}\mathcal{F} := R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{F}, \mathcal{D}_X)[\dim X] \in D^b_{coh}(\mathcal{M}^r(\mathcal{D}_X))$$

Similarly, we define the dual of any right $\mathcal{D}_X$-module.

Some time ago, we showed that if $\mathcal{F}$ is a single module then $\mathcal{E}xt^i(\mathcal{F}, \Pi)$ vanishes for $i > \dim X$. This implies that the dual complex is indeed bounded.

Theorem 15.3. $\mathbb{D}^2 \cong \mathrm{Id}$

Proof. First of all let us describe the canonical map $\mathrm{Id} \to \mathbb{D}^2$. Let $\mathcal{F} \in D^b_{coh}(\mathcal{M}(\mathcal{D}_X))$ and let $\mathcal{H} := \mathbb{D}\mathcal{F}[-\dim X] = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{F}, \mathcal{D}_X)$. Then we have a canonical map $\mathcal{F} \boxtimes \mathcal{H} = \mathcal{F} \otimes_{\mathbb{C}} \mathcal{H} \to \mathcal{D}_X$. Now, $\mathrm{Hom} = \mathsf{H}^0(R\Gamma(R\mathcal{H}om))$, and

$$R\mathcal{H}om_{\mathcal{D}_X \boxtimes \mathcal{D}_X}(\mathcal{F} \boxtimes \mathcal{H}, \mathcal{D}_X) \cong R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{F}, R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{H}, \mathcal{D}_X))$$

Thus $\mathrm{Hom}_{\mathcal{D}_X \boxtimes \mathcal{D}_X}(\mathcal{F} \boxtimes \mathcal{H}, \mathcal{D}_X) \cong \mathrm{Hom}(\mathcal{F}, R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{H}, \mathcal{D}_X))$, and thus the canonical map $\mathcal{F} \boxtimes \mathcal{H} \to \mathcal{D}_X$ gives a map $\mathcal{F} \to R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{H}, \mathcal{D}_X) \cong \mathbb{D}^2\mathcal{F}$. In order to show that this map is an isomorphism, it is enough to show this in the case of affine X, and $\mathcal{F} \cong \mathcal{D}_X$. This case is obvious. $\square$

Lemma 15.4. $\mathbb{D}(\mathcal{O}_X) \cong \omega_X$.

Proof. We have to show that $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{D}_X)[n] \cong \omega_X$.

Recall the locally projective resolution of $\mathcal{O}_X$ given by

$$0 \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \Lambda^n(\tau_X) \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \Lambda^{n-1}(\tau_X) \to \cdots \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \tau_X \to \mathcal{D}_X \to 0$$

Note that

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \Lambda^i(\tau_X), \mathcal{D}_X) \cong \mathcal{H}om_{\mathcal{O}_X}(\Lambda^i(\tau_X), \mathcal{D}_X) \cong \Omega^i_X \otimes_{\mathcal{O}_X} \mathcal{D}_X$$

Thus, $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{D}_X)[n]$ is isomorphic to the de-Rham resolution of ω_X, and thus to ω_X. $\square$

In the same way one proves

Lemma 15.5. *Let M be a smooth $\mathcal{D}_X$-module. Then $\mathbb{D}M \cong \mathcal{H}om_{\mathcal{O}_X}(M, \omega_X)$ as an $\mathcal{O}_X$-module.*

In the proof we use the fact that every smooth $\mathcal{D}_X$-module is locally free over $\mathcal{O}_X$ and thus on can just tensor the resolution of $\mathcal{O}_X$ with M over $\mathcal{O}_X$ to obtain a resolution of M.

Exercise 15.6. For $\mathcal{F}, \mathcal{H} \in D^b_{coh}(\mathcal{M}(\mathcal{D}_X))$, $\mathcal{H}om(\mathcal{F}, \mathcal{H}) \cong \mathbb{D}(\mathcal{F}) \otimes^L \mathcal{H}[-\dim X]$.

Proposition 15.7. *For projective morphisms $\pi : X \to Y$, $\pi_*\mathbb{D} \cong \mathbb{D}\pi_*$ on $D^b_{coh}(\mathcal{M}(\mathcal{D}_X))$.*

Proof. By the Yoneda lemma, it is enough to show that

$$\mathrm{Hom}_{\mathcal{D}_Y}(\pi_*\mathbb{D}\mathcal{F},\mathcal{H}) \cong \mathrm{Hom}_{\mathcal{D}_Y}(\mathbb{D}\pi_*\mathcal{F},\mathcal{H}).$$

Since π_* is adjoint to $\pi^!$ for projective π, we have

$$\mathrm{Hom}_{\mathcal{D}_Y}(\pi_*\mathbb{D}\mathcal{F},\mathcal{H}) \cong \mathrm{Hom}_{\mathcal{D}_X}(\mathbb{D}\mathcal{F},\pi^!\mathcal{H}).$$

Since $\mathrm{Hom} = \mathsf{H}^0(R\Gamma(\mathcal{H}om))$, it is enough to show that

$$R\pi_\bullet(\mathcal{H}om(\mathbb{D}\mathcal{F},\pi^!\mathcal{H})) \cong \mathcal{H}om(\mathbb{D}\pi_*\mathcal{F},\mathcal{H})$$

By the previous exercise,

$\mathcal{H}om(\mathbb{D}\mathcal{F},\pi^!\mathcal{H}) \cong \mathcal{F}\otimes^L_{\mathcal{D}_X}\pi^!\mathcal{H}[-\dim X]$ and $\mathcal{H}om(\mathbb{D}\pi_*\mathcal{F},\mathcal{H}) \cong \pi_*\mathcal{F}\otimes^L_{\mathcal{D}_Y}\mathcal{H}[-\dim Y]$.

Further,

$$\begin{aligned}\pi_*\mathcal{F}\otimes^L_{\mathcal{D}_Y}\mathcal{H}[-\dim Y] &\cong R\pi_\bullet(\mathcal{F}\otimes^L_{\mathcal{D}_Y}\mathcal{D}_{X\to Y})\otimes^L_{\mathcal{D}_Y}\mathcal{H}[-\dim Y]\cong \\ &\cong R\pi_\bullet(\mathcal{F}\otimes^L_{\mathcal{D}_X}\pi^\bullet(\mathcal{H})\otimes^L_{\mathcal{D}_Y}\mathcal{D}_{X\to Y})[-\dim Y]\cong R\pi_\bullet(\mathcal{F}\otimes^L_{\mathcal{D}_X}\pi^!\mathcal{H})[-\dim X]\end{aligned}$$

Altogether,

$$R\pi_\bullet(\mathcal{H}om(\mathbb{D}\mathcal{F},\pi^!\mathcal{H}))\cong R\pi_\bullet(\mathcal{F}\otimes^L_{\mathcal{D}_X}\pi^!\mathcal{F})[-\dim X]\cong \pi_*\mathcal{F}\otimes^L_{\mathcal{D}_Y}\mathcal{H}[-\dim Y]\cong \mathcal{H}om(\mathbb{D}\pi_*\mathcal{F},\mathcal{H})$$

as required. □

We can use this proposition to define $\mathbb{D}$ on singular varieties: we define the functor locally, and on affine variety we do this by embedding into an affine space.

Some time ago, we proved the following theorem (for smooth X).

Theorem 15.8. *For any $\mathcal{F}\in\mathcal{M}_{coh}(\mathcal{D}_X)$ we have*

(i) $\mathcal{E}xt^i(\mathcal{F},\mathcal{D}_X)=0 \quad \forall i < codim\,\mathrm{SingSupp}(\mathcal{F})$

(ii) $\mathrm{codim}\,\mathrm{SingSupp}(\mathcal{E}xt^i(\mathcal{F},\mathcal{D}_X))\geq i \quad \forall i$

Corollary 15.9. *For any $\mathcal{F}\in\mathcal{M}_{coh}(\mathcal{D}_X)$, the following are equivalent:*

$$\mathbb{D}\mathcal{F}\in\mathcal{M}_{coh}(\mathcal{D}_X)\Leftrightarrow \mathcal{F} \text{ is holonomic } \Leftrightarrow \mathbb{D}\mathcal{F} \text{ is holonomic}$$

Finally, let us show that $\mathbb{D}$ preserves singular support for all complexes of modules.

Exercise 15.10. Let (A,F) be an algebra with a good filtration such that $\mathrm{Gr}_F A$ is commutative, and let (M,Φ) be a good filtered module. Then for any i there exists a good filtration on $\mathcal{E}xt^i(M,A)$ such that $\mathrm{Gr}\,\mathcal{E}xt^i(M,A)$ is a subquotient of $\mathcal{E}xt^i(\mathrm{Gr}\,M,\mathrm{Gr}\,A)$.

Corollary 15.11. $\mathrm{Supp}\,\mathrm{Gr}\,\mathcal{E}xt^i(M,A)\subset\mathrm{Supp}\,\mathrm{Gr}\,M$

Definition 15.12. The singular support of a bounded complex of (sheaves of) $\mathcal{D}_X$-modules is the union of singular supports of the cohomologies.

Corollary 15.13. *For any $\mathcal{F}\in D^b_{coh}(\mathcal{M}(\mathcal{D}_X))$, we have* $\mathrm{SingSupp}\mathbb{D}\mathcal{F}=\mathrm{SingSupp}\mathcal{F}$.

Proof. Since $\mathbb{D}^2\cong\mathrm{Id}$, it is enough to prove that $\mathrm{SingSupp}\mathbb{D}\mathcal{F}\subset\mathrm{SingSupp}\mathcal{F}$. Since $\mathbb{D}$ is an equivalence, it preserves exact triangles. For $\mathcal{F}\in D^{[l,m]}$ we have the triangle

$$\mathsf{H}^l(\mathcal{F})\to\mathcal{F}\to\tau^{>l}\mathcal{F}$$

Using this triangle, and induction on $m-l$, we can assume that $\mathcal{F}$ is a single module. Localizing, we can assume that X is affine. For affine X it is the previous corollary. □

16. Preservation of holonomicity

In this section we show that the functors defined above preserve holonomicity. For the duality we already proved this.

Theorem 16.1. *Pushforward preserves holonomicity.*

Proof. For closed embeddings this holds since they are affine maps, and since pushforward is an equivalence of categories for them.

For projections $\mathbb{P}^n \to pt$ pushforward sends all coherent complexes to complexes with finite-dimensional cohomologies.

Thus it is enough to prove for open embeddings $j : U \hookrightarrow X$. Since the statement is local on X, we can assume that X is affine. Let $U = \bigcup U_i$ be a cover of U by basic open subsets. Let $j_i : U_i \hookrightarrow X$ be inclusions, and $\mathcal{F}$ be a holonomic $\mathcal{D}_U$-module. Then $j_0\mathcal{F}$ embeds into $\bigoplus (j_i)_0(\mathcal{F}_{U_i})$, which is holonomic as we showed some time ago. Now, j_* of a complex is obtained by applying j_0 elementwise. □

Remark 16.2. The proof for open affine embeddings that we gave some time ago is based on the case of affine spaces. For them we used Fourier transform and Bernstein filtration, and in some sense this was a trick. In particular, this trick does not work in the parallel realm of analytic $\mathcal{D}$-modules. Thus, Kashiwara invented an alternative proof. I write it in the end of the section, and we will go over it if we will have time.

Theorem 16.3 (Bernstein)**.** *The natural embedding $D^b(\mathcal{H}ol(\mathcal{D}_X)) \hookrightarrow D^b_{\mathcal{H}ol}(\mathcal{M}(\mathcal{D}_X))$ is an equivalence of categories.*

We will neither use nor prove this theorem.

16.1. **Holonomicity can be checked fiberwise, and thus is preserved by pullbacks.**

We will use the following lemma from algebraic geometry.

Lemma 16.4. *For any morphism $\pi : W \to X$ and any coherent $\mathcal{O}_W$-module, there exists an open subset $U \subset X$ such that $\pi_\bullet(\mathcal{F}|_{\pi^{-1}(U)})$ is $\mathcal{O}_U$-free.*

Corollary 16.5. *For any coherent $\mathcal{D}_X$-module $\mathcal{F}$, there exists an open subset $U \subset X$ such that $\mathcal{F}|_U$ is $\mathcal{O}_U$-free.*

Proof. $\operatorname{Gr}\mathcal{F}$ is a coherent $\mathcal{O}_{T^*X}$-module. Now we use the previous lemma for $\pi : T^*X \to X$. We have $\mathcal{F}|_U = \pi_\bullet(\operatorname{Gr}\mathcal{F}|_{\pi^{-1}(U)})$ as an $\mathcal{O}_U$-module. □

Proposition 16.6 (Fiberwise holonomicity criterion)**.** *A complex $\mathcal{F} \in D^b_{coh}(\mathcal{M}(\mathcal{D}_X))$ is holonomic if and only if the fiber $i_x^!\mathcal{F}$ has finite-dimensional cohomologies for every point $x \in X$.*

Proof. First suppose that $\mathcal{F}$ is holonomic and let $x \in X$. Let j denote the open embedding $j : X \setminus \{x\} \hookrightarrow X$. Then we have the distinguished triangle

$$(i_x)_* i_x^! \mathcal{F} \to \mathcal{F} \to j_*(\mathcal{F}|_{X\setminus\{x\}})$$

The pushforward $j_*(\mathcal{F}|_{X\setminus\{x\}})$ is holonomic, and cone of holonomic complexes is holonomic. Thus $(i_x)_* i_x^!\mathcal{F}$ is holonomic and thus so is $i_x^!\mathcal{F}$.

To the other direction, let $\mathcal{F} \in \mathcal{M}_{coh}(\mathcal{D}_X)$ be a coherent module such that all fibers $i_x^!\mathcal{F}$ have finite-dimensional cohomologies. We want to prove that $\mathcal{F}$ is holonomic. It is enough to prove for the case of a single module since every complex is glued from its cohomologies.

By the previous corollary, there exists $U \subset X$ such that $\mathcal{F}|_U$ is $\mathcal{O}_U$-free, and thus flat. Thus the fiber is concentrated in the zero's cohomology, and $\mathcal{F}|_U$ is $\mathcal{O}$-coherent (since the fiber of $\mathcal{O}^\alpha$ is K^α. But $\mathcal{O}$-coherent modules are holonomic, so $\mathcal{F}|_U$ is holonomic. Let $Z := X \setminus U$. We can assume that the theorem holds for Z by Noetherian induction. The fibers of $i_Z^!\mathcal{F}$ are the fibers of $\mathcal{F}$ in points of Z, and thus have finite-dimensional cohomologies. Thus $i_Z^!\mathcal{F}$ is holonomic. Now, we have the exact triangle

$$(i_Z)_* i_Z^! \mathcal{F} \to \mathcal{F} \to (j_U)_* j_U^! \mathcal{F}$$

and pushforward preserves holonomicity. Thus $\mathcal{F}$ is holonomic. □

Corollary 16.7. *Let $\pi : X \to Y$ be a morphism of algebraic varieties, and $\mathcal{F} \in D^b_{Hol}(\mathcal{M}(\mathcal{D}_Y))$. Then $\pi^!\mathcal{F} \in D^b_{Hol}$.*

Proof. For every $x \in X$, $i_x^! \pi^! \mathcal{F} \cong i_{\pi(x)}^! \mathcal{F}$. Thus the statement follows from the previous proposition. □

Altogether we obtain

Corollary 16.8. *The functors $\pi^!, \pi_*$, and $\mathbb{D}$ preserve holonomicity.*

Remark 16.9. Let us give two negative examples so that you appreciate the fiberwise holonomicity criterion. First of all, some fibers of non-holonomic coherent modules have infinite-dimensional cohomologies. Next, all fibers of coherent $\mathcal{O}$-modules are finite-dimensional, but there exist non-coherent $\mathcal{O}$-modules with all fibers vanishing. For example, $M = \mathbb{K}(t)$ over $X = \mathbb{A}^1$.

16.2. **6 functors of Grothendieck.** In this subsection we consider only holonomic complexes, and define two new functors:

$$\pi^* := \mathbb{D}\pi^!\mathbb{D} \text{ and } \pi_! := \mathbb{D}\pi_*\mathbb{D}.$$

Why do we only define these for holonomic modules? Because $\mathbb{D}^2 \cong \mathrm{Id}$ only for coherent modules, and π_* and $\pi^!$ do not preserve coherence in general.

Proposition 16.10. *π^* is left adjoint to π_* and $\pi_!$ is left adjoint to $\pi^!$.*

Proof. We will prove this for $\pi^!$ and $\pi_!$, since for the other pair this is similar. Since $\mathrm{Hom} \cong \mathsf{H}^0(R\mathcal{H}om)$, it is enough to show that

$$R\pi_\bullet R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{F}, \pi^!\mathcal{H}) \cong R\mathcal{H}om_{\mathcal{D}_Y}(\pi_!\mathcal{F}, \mathcal{H})$$

We have

$$R\pi_\bullet R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{F}, \pi^!\mathcal{H}) \cong R\pi_\bullet(\mathbb{D}\mathcal{F} \otimes^L_{\mathcal{D}_X} \pi^!\mathcal{H}))[-\dim X] \cong$$
$$R\pi_\bullet(\mathbb{D}\mathcal{F} \otimes^L_{\mathcal{D}_X} \mathcal{D}_{X\to Y} \otimes^L_{\pi^\bullet\mathcal{D}_Y} \pi^\bullet\mathcal{H})[-\dim Y]) \cong R\pi_\bullet(\mathbb{D}\mathcal{F} \otimes^L_{\mathcal{D}_X} \mathcal{D}_{X\to Y}) \otimes^L_{\mathcal{D}_Y} \mathcal{H})[-\dim Y] \cong$$
$$\cong \pi_*\mathbb{D}\mathcal{F} \otimes^L_{\mathcal{D}_Y} \mathcal{H}[-\dim Y] \cong R\mathcal{H}om(\mathcal{F}, \mathcal{H})$$

□

This again implies that for projective morphisms, $\pi_* \cong \pi_!$, after we identify left modules with right ones. We actually already showed this for all coherent modules in Proposition 15.7.

Thus, for projective π, π^* and $\pi^!$ are left resp. right adjoint to the same functor on two different sides. Similarly, for an open embedding $j : U \hookrightarrow X$, $j_!$ and j_* are left and right adjoint functors to the restriction $\mathcal{F} \to \mathcal{F}|_U$.

Exercise 16.11. Let $p : X \to pt$, and let $\mathcal{F}, \mathcal{H} \in D^b_{Hol}(\mathcal{M}(\mathcal{D}_X))$. Then

$$\mathrm{RHom}_{\mathcal{D}_X}(\mathcal{F}, \mathcal{H}) \cong p_*(\mathbb{D}\mathcal{H} \otimes^! \mathcal{F}).$$

Hint. Since $\mathrm{RHom}_{\mathcal{D}_X}(\mathcal{F}, \mathcal{H}) \cong R\Gamma(R\mathcal{H}om(\mathcal{F}, \mathcal{H}))$, and

$$R\mathcal{H}om(\mathcal{F}, \mathcal{H}) \cong \mathbb{D}\mathcal{F} \otimes^L_{\mathcal{D}_X} \mathcal{H}[-\dim X],$$

it is enough to show that $\mathbb{D}\mathcal{F} \otimes^L_{\mathcal{D}_X} \mathcal{H}[-\dim X] \cong (\mathbb{D}\mathcal{F} \otimes^! \mathcal{H}) \otimes^L_{\mathcal{D}_X} \mathcal{O}_X$. $\square$

16.3. **Functors given by kernels.** This subsection refers to general modules, not necessarily holonomic. Let X and Y be algebraic varieties. Any $K \in D^b(\mathcal{D}_{X\times Y})$ defines a functor $D^b(\mathcal{D}_X) \to D^b(\mathcal{D}_Y)$ by

$$T_K(\mathcal{F}) := (\pi_Y)_*(K \otimes^! \pi^!_X(\mathcal{F})),$$

where $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are projections. I remind that $\cdot \otimes^! \cdot = \Delta^!(\cdot \boxtimes \cdot)$.

This mimics the way matrix define linear operators: $\mathcal{F}$ can be viewed as a vector $v = (v_i)$, $\pi^!_X\mathcal{F}$ is viewed a matrix given by $B_{ij} = v_j$, K is a matrix, tensor product corresponds to elementwise multiplication: $C_{ij} = K_{ij}B_{ij} = K_{ij}v_j$, and $(\pi_Y)_*$ is parallel to summation over Y: $\sum_j C_{ij} = \sum_j K_{ij}v_j$.

Proposition 16.12. *(i) ν_* is given by the kernel $(\Gamma_\nu)_*\mathcal{O}_X$, where $\Gamma_\nu : X \to X \times Y$ is the graph of ν: $\Gamma_\nu(x) = (x, \nu(x))$.*

(ii) $\nu^!$ is given by the kernel $(\Gamma'_\nu)_*\mathcal{O}_X$, where $\Gamma'_\nu : X \to Y \times X$ is given by $\Gamma'_\nu(x) = (\nu(x), x)$.

(iii) $T_{K_1}T_{K_2} = T_{K_1 * K_2}$, where $K_1 * K_2 = (\pi_{X\times Z})_*(\Delta^!_Y(K_1 \boxtimes K_2))$

Note that (iii) mimics matrix multiplication.

For the proof we will need the following lemma.

Lemma 16.13. *Let $\nu : X \to Y$ be a morphism of algebraic varieties and let $\mathcal{F} \in D^b(\mathcal{M}(\mathcal{D}_X))$ and $\mathcal{H} \in D^b(\mathcal{M}(\mathcal{D}_Y))$. Then*

$$\nu_*(\mathcal{F} \otimes^! \nu^!\mathcal{H}) \cong \nu_*\mathcal{F} \otimes^! \mathcal{H}.$$

Proof. Decomposing $\Gamma_\nu : X \hookrightarrow X \times Y$ into $X \xrightarrow{\Delta} X \times X \xrightarrow{\mathrm{Id}\times\nu} X \times Y$, we get that

$$\mathcal{F} \otimes^! \nu^!\mathcal{H} \cong \Gamma^!_\nu(\mathcal{F} \boxtimes \mathcal{H}).$$

The lemma follows now from the base change theorem applied to the following Cartesian square:

$$\begin{tikzcd} X^{\mathcal{F}\otimes^! \nu^! \mathcal{H}} \arrow[r, "\Gamma_\nu"] \arrow[d, "\nu"] & X \times Y^{\mathcal{F}\boxtimes\mathcal{H}} \arrow[d, "\nu\times\mathrm{Id}"] \\ Y^{\nu_*\mathcal{F}\otimes^!\mathcal{H}} \arrow[r, "\Delta"] & Y \times Y^{\nu_*\mathcal{F}\boxtimes\mathcal{H}} \end{tikzcd}$$

□

Exercise 16.14. For any $\mathcal{D}_X$-module $\mathcal{F}$, $\mathcal{O}_X \otimes^! \mathcal{F} \cong \mathcal{F}$.

Proof of Proposition 16.12(i).

$$(\pi_Y)_*((\Gamma_\nu)_*\mathcal{O}_X \otimes^! \pi_X^! \mathcal{F}) \cong (\pi_Y)_*(\Gamma_\nu)_*(\mathcal{O}_X \otimes^! \Gamma_\nu^! \pi_X^! \mathcal{F}) \cong \nu_*(\mathcal{O}_X \otimes^! \mathcal{F}) \cong \nu_*\mathcal{F}$$

□

Exercise 16.15. Prove the rest of the proposition.

16.4. **Radon transform.** Motivation: let F be a finite field, and V be a finite-dimensional vector space over F. Let $X := \mathbb{P}(V)$ and $X' := \mathbb{P}(V^*)$. Let $F(X)$ denote the space of all functions $X \to F$. For any $y \in X'$ denote by $H_y \subset X$ the hyperplane on which y vanishes.

Define $R, \tilde{R} : F(X) \to F(X')$ by $Rf(y) = \sum_{x\in H_y} f(x)$ and $\tilde{R}f(y) := \sum_{x\notin H_y} f(x)$. To describe the kernels of these operators let

$$I = \{x \in X, y \in X' \,|\, x \in H_y\}.$$

Then the kernel of R is the characteristic function χ_I of I, and the kernel of $\tilde{R}$ is $1-\chi_I$.

Going back to $\mathcal{D}$-modules we let V be a vector space over $\mathbb{K}$, $X = \mathbb{P}(V)$, $X' = \mathbb{P}(V^*)$, and I be as above. Let $U := X \times X' \setminus I$ and let j denote the embedding $j : U \hookrightarrow X \times X'$.

Define $\tilde{R} : D^b(\mathcal{M}(\mathcal{D}_X)) \to D^b(\mathcal{M}(\mathcal{D}'_X))$ by the kernel $j_*\mathcal{O}_U$.

Exercise 16.16. The functor $\tilde{R}$ is an equivalence of categories, with pseudo-inverse given by $j_!\mathcal{O}_{U'}$, where $U' \subset X' \times X$ is the subset corresponding to U.

Hint. Show that $K := j_*\mathcal{O}_U * j_!\mathcal{O}_U \cong (\Delta_X)_*\mathcal{O}_X$ in two steps:
1. $\forall x \neq y \in X$, $i^!_{(x,y)}(K) = 0$, and thus K is supported on Δ_X. 2. $\Delta_X^! K \cong \mathcal{O}_X$. □

Here we use

Exercise 16.17. Every non-zero coherent $\mathcal{D}$-module has a non-zero fiber.

16.5. **Kashiwara's proof that pushforward under open affine embedding preserves holonomicity.** We will first prove some properties of holonomic modules.

Lemma 16.18. *Holonomic modules have finite length.*

Proof. Let $\mathcal{F} \in \mathcal{H}ol(\mathcal{D}_X)$. Then $\mathcal{F}$ is Noetherian. But $\mathbb{D}(\mathcal{F})$ is also holonomic, thus also Noetherian, thus $\mathcal{F} \cong \mathbb{D}(\mathbb{D}((\mathcal{F}))$ is Artinian, thus finite length. □

Lemma 16.19. *Let $\mathcal{F} \in \mathcal{M}(\mathcal{D}_X)$, let $U \subset X$ be an open subset, and let $\mathcal{H} \subset \mathcal{F}|_U$ be a holonomic submodule. Then there exists a holonomic $\mathcal{H}' \subset \mathcal{F}$ such that $\mathcal{H}'|_U = \mathcal{H}$.*

Proof. By a similar lemma on coherent modules proven before we can assume that $\mathcal{F}$ is coherent. Consider $\mathbb{D}\mathcal{F}$ - a complex concentrated in degrees ≤ 0. Let $Q := \tau_{\geq 0}\mathbb{D}\mathcal{F}$ - a single module. We have a map $\mathbb{D}\mathcal{F} \to Q$, and its dual gives $\mathbb{D}Q \to \mathbb{D}^2\mathcal{F} \cong \mathcal{F}$. Then the image $\mathcal{G} \subset \mathcal{F}$ of this map is the maximal holonomic submodule of $\mathcal{F}$. This construction is compatible with restrictions to open subsets, since $\tau_{\geq 0}$ and $\mathbb{D}$ are local operations. Thus, $\mathcal{G}|_U$ is the maximal holonomic submodule of $\mathcal{F}|_U$, and thus includes $\mathcal{H}$. Let $\mathcal{H}'' \subset \mathcal{G}$ be the preimage of $\mathcal{H}$ under the restriction $\mathcal{G} \to \mathcal{G}|_U$. □

Now let X be affine, and U is a basic open subset: $U = X_f = \{x \in X \mid f(x) \neq 0\}$, for some polynomial $f \in \mathcal{O}(X)$. For any $\mathcal{D}(X_f)$-module M, $j_*M = j_0M = M$, viewed as a $\mathcal{D}(X)$-module. First of all we need to show that M is finite-generated. For this we need to obtain $f^{-k-1}u$ from $f^{-k}u$ for some $k \in \mathbb{Z}$ and some generator u of M (holonomic modules are cyclic). We have already done such a thing, when proving analytic continuation of p^λ.

Lemma 16.20. *Let $M = \langle u \rangle$ be a holonomic $\mathcal{D}(U)$-module. Then there exist $d \in \mathcal{D}(X)[\lambda]$ and $b \in \mathbb{K}[\lambda]$ such that $d(f^{\lambda+1}u) = bf^\lambda u$, where λ is a formal parameter.*

Proof. Let $\widetilde{D} := \mathcal{D}(X)[\lambda]$, $\widetilde{D_f} = \mathcal{D}(U)[\lambda]$, $\widetilde{R} := R[\lambda]$. Consider new $\widetilde{D_f}$-module $Q := \widetilde{R}f^\lambda$. It consists of elements of the form rf^λ with $r \in \widetilde{R}$ with the action of vector fields given by

$$\xi r f^\lambda := (\xi(r) + \lambda r f^{-1}\xi(f))f^\lambda$$

Extend scalars to $\mathbb{K}(\lambda)$, i.e. tensor with it over $\mathbb{K}[\lambda]$

Step 1 $Q_{\mathbb{K}(\lambda)}$ is a holonomic $D(U)_{K(\lambda)}$-module.

Pf: Consider the automorphism of $\mathcal{D}(U)_{K(\lambda)}$ given by $\tau(g) := g, \tau(\xi) := \xi + \lambda f^{-1}\xi(f)$. Then $\tau R_{K(\lambda)} \cong Q_{K(\lambda)}$, and thus $Q_{K(\lambda)}$ is holonomic.

Step 2 $\exists\, d' \in \mathcal{D}(X)_{K(\lambda)}$ s.t. $d'(uff^\lambda) = uf^\lambda$.

Pf: $Q = \langle uf^\lambda \rangle$, so $j_0Q_{K(\lambda)} = \langle uf^k f^\lambda,\ k \in \mathbb{Z} \rangle$. Thus $j_0Q_{K(\lambda)}|_U$ is holonomic, thus by the holonomic extension lemma (Lemma 16.19) there exists a submodule $H \subset j_0Q_{\mathbb{K}(\lambda)}$ s.t. $H|_U = Q_{\mathbb{K}(\lambda)} = j_0Q_{\mathbb{K}(\lambda)}|_U$. Since $(j_0Q_{\mathbb{K}(\lambda)}/H)|_U = 0$, any element of this quotient is annihilated by some power of f. Let $\tilde{u} := uf^\lambda \in j_0Q_{\mathbb{K}(\lambda)}$. Then $f^k\tilde{u} \in H$. For any $i \geq 0$, let $M_i \subset H$ be the submodule generated by $f^{i+k}\tilde{u}$. Since H is holonomic, it has finite length and thus the sequence M_i stabilizes. Thus there exists $d'' \in D_{K(\lambda)}$ such that

$$f^{i+k}uf^\lambda = d''(f^{i+k+1}uf^\lambda)$$

Conjugating d'' by the automorphism $\lambda \to \lambda + k$ we get d' s.t.

$$uf^\lambda = d'(uf^{\lambda+1})$$

Finally, write $d' = b^{-1}d$ with $b \in \mathbb{K}[\lambda]$ and $d \in \mathcal{D}(X)[\lambda]$.

□

Corollary 16.21. *M is holonomic as a $\mathcal{D}_X$-module.*

Proof. By the previous lemma, M is generated by $f^k u$ for some $k \in \mathbb{Z}$. Consider the decreasing sequence of submodules $M_i := \langle f^{k+i}u \rangle$. For i big enough, M_i are holonomic by the extension lemma (Lemma 16.19). Consider $\widetilde{D} := \mathcal{D}(X)[\lambda]$. Then

$$\mathrm{Spec}(\mathrm{Gr}(\widetilde{D})) = T^*X \times \mathbb{A}^1.$$

We have

$$\widetilde{\Delta} := \mathrm{SingSupp}(M[\lambda]) \subset T^*X \times \mathbb{A}^1,$$

and

$$\mathrm{SingSupp}M_i = \widetilde{\Delta_i} := \widetilde{\Delta} \cap (T^*X \times \{k+i\}).$$

Under $\tau : \lambda \mapsto \lambda + 1$, $\tau M[\lambda] \subset M[\lambda]$. Thus $\widetilde{\Delta}$ is τ-invariant, and thus $\widetilde{\Delta} = \Delta \times \mathbb{A}^1$. Thus all M_i are holonomic, including $M = M_0$. $\square$

17. Perverse extensions and classification of simple holonomic modules

Let $j : U \hookrightarrow X$ be an open embedding, and $\mathcal{F}$ be a holonomic $\mathcal{D}_U$-module. Then we have a canonical map $\varphi_{\mathcal{F}} : j_!\mathcal{F} \to j_*\mathcal{F}$ (where we define j_* using identification of left and right $\mathcal{D}$-modules). Indeed,

$$\mathrm{Hom}(j_!\mathcal{F}, j_*\mathcal{F}) \cong \mathrm{Hom}(\mathcal{F}, (j_*\mathcal{F})|_U) \cong \mathrm{Hom}(\mathcal{F}, \mathcal{F}) \ni \mathrm{Id}$$

The cone of $\varphi_{\mathcal{F}}$ is supported on $Z := X \setminus U$. In some sense it is the limit of φ at Z.

If U is affine then both $j_*\mathcal{F} = j_0\mathcal{F}$ and $j_!\mathcal{F}$ are single $\mathcal{D}_X$ - modules, then the image of $\varphi_{\mathcal{F}}$ is also a single module. It is called the perverse extension of $\mathcal{F}$ and denoted by $j_{!*}\mathcal{F}$. It is the minimal submodule of $j_*\mathcal{F}$ whose restriction to U is $\mathcal{F}$. Indeed, for any $\mathcal{H} \subset \mathcal{F}$ with $\mathcal{H}|_U = \mathcal{F}$ we have from adjunction of $j_!$ and $j^!$ =restriction a map $j_!\mathcal{F} \to \mathcal{H}$. Since the canonical map $\varphi_{\mathcal{F}} : j_!\mathcal{F} \to j_*\mathcal{F}$ is obtained in the same way, it factors as $j_!\mathcal{F} \to \mathcal{H} \subset j_*\mathcal{F}$, and thus $\mathcal{H}$ includes the image $j_{!*}$ of φ.

If U is not affine then $j_*\mathcal{F} = Rj_0\mathcal{F} \in \mathcal{D}^{\geq 0}_{hol}(\mathcal{M}(\mathcal{D}_X))$, i.e. it is supported in non-negative indices. On the other hand, $j_!\mathcal{F} \in \mathcal{D}^{\leq 0}_{hol}(\mathcal{M}(\mathcal{D}_X))$. Thus the map $j_!\mathcal{F} \to j_*\mathcal{F}$ factors as

$$j_!\mathcal{F} \to \tau_{\geq 0} j_!\mathcal{F} \to \tau_{\leq 0} j_*\mathcal{F} \to j_*\mathcal{F}$$

Here, $\tau_{\geq 0} j_!\mathcal{F}$ and $\tau_{\leq 0} j_*\mathcal{F}$ are single modules, and we define $j_{*!}\mathcal{F}$ as the image of the map $\tau_{\geq 0} j_!\mathcal{F} \to \tau_{\leq 0} j_*\mathcal{F}$.

Example 17.1. Let $j : U = \mathbb{A}^1 \setminus \{0\} \hookrightarrow \mathbb{A}^1$, and $\mathcal{F} = \mathcal{O}_U = L := \mathbb{K}[x, x^{-1}]$ - Laurent polynomials. Then $j_*\mathcal{F} = L$ as $\mathcal{D}_X = \mathcal{D}_1$-module. It is generated by x^{-1}. We have

$$0 \to \mathcal{O} \to L \to \Delta \to 0,$$

where $\Delta = \mathbb{K}[x, x^{-1}]/\mathbb{K}[x]$, and it is isomorphic to the $\mathcal{D}_X$-module generated by Dirac's δ-function. Since $\mathcal{O}$ and Δ are self-dual, we get that $j_!\mathcal{O}_U \cong \mathbb{D}L$ and we have the short exact sequence

$$0 \to \Delta \to \mathbb{D}L \to \mathcal{O} \to 0.$$

Thus, $j_{!*}\mathcal{F} \cong \mathcal{O}_X$ and $\mathrm{Cone}(\varphi_{\mathcal{F}})$ is glued from Δ and $\Delta[-1]$ (or $\Delta[-1]$?).

For locally closed embeddings $j : W \hookrightarrow U \hookrightarrow X$, with $U \subset X$ open, and W a closed subset of U, we define $j_{!*} := (j_U)_{!*} \circ (i_W)_*$.

Apparently, any simple holonomic module is a perverse extension of a smooth D-module from a locally closed subvariety.

Theorem 17.2. *Let X be a (quasi-projective) algebraic variety.*

(i) *Let W be a locally-closed subvariety such that $j : W \to X$ is affine, and let $\mathcal{F}$ be a simple holonomic $\mathcal{D}_W$-module. Then $L(W, \mathcal{F}) := j_{!*}\mathcal{F}$ is also simple, and is the unique simple submodule of $j_*\mathcal{F}$ and the unique simple quotient of $j_!\mathcal{F}$.*

(ii) *Any simple holonomic $\mathcal{D}_X$-module is isomorphic to a module of the form $L(W, \mathcal{F})$ for some locally-closed W and some* smooth *$\mathcal{D}_W$-module $\mathcal{F}$.*

(iii) $L(W, \mathcal{F}) \simeq L(W', \mathcal{F}') \iff \overline{W} = \overline{W'}$ *and* $\mathcal{F}|_{W\cap W'} \cong \mathcal{F}'|_{W\cap W'}$

Proof. (i) We can assume that X is affine, and $W = U \subset X$ is open affine. Let $S \subset j_*\mathcal{F}$ be a simple submodule. Then $S|_W \subset j_*\mathcal{F}|_W = j_!\mathcal{F}|_W = \mathcal{F}$. Thus $S \cap j_{!*}\mathcal{F} \neq 0$ and since S is simple we have

$$0 \to S \to j_{!*}\mathcal{F} \to C \to 0$$

for some C supported in $Z := X \setminus W$. But $j_!\mathcal{F}$ has no quotients supported at Z. Indeed,

$$\mathrm{Hom}(j_!\mathcal{F}, C) \cong \mathrm{Hom}(\mathcal{F}, j^!C) \cong \mathrm{Hom}(\mathcal{F}, C|_U) = 0$$

Thus $C = 0$ and thus $S = j_{!*}\mathcal{F}$ and thus $j_{!*}\mathcal{F}$ is simple.

(ii) Let $\mathcal{H}$ be a simple holonomic $\mathcal{D}_X$-module, and let Z be an irreducible component of Supp$\mathcal{F}$, and Z' be an open dense smooth subvariety. As we showed before, there exists an open dense subset $W \subset Z'$ such that $\mathcal{F}|_W$ is smooth (i.e. $\mathcal{O}_W$-coherent). Let $\mathcal{F} := j_W^!\mathcal{H}$. Let $\mathcal{F}' \subset \mathcal{F}$ be a simple submodule. From the inclusion $\mathcal{F}' \hookrightarrow \mathcal{F} = j_W^!\mathcal{H}$, we get a map $(j_W)_!\mathcal{F}' \to \mathcal{H}$.

We claim that this map factors through $(j_W)_{!*}\mathcal{F}'$. To show this, we can assume that $Z = X$ and thus W is open in X. In this case we have maps $j_!\mathcal{F}' \to \mathcal{H} \to j_*\mathcal{F}'$, and their composition is the map $j_!\mathcal{F} \to j_*\mathcal{F}$. Since $\mathcal{H}$ is simple, the map $\mathcal{H} \to j_*\mathcal{F}'$ is an isomorphism to its image, which in turn equals $j_{!*}\mathcal{F}'$. Thus, we get a non-zero map $(j_W)_{!*}\mathcal{F}' \to \mathcal{H}$ in general. It is an isomorphism since both modules are simple.

(iii) Since Supp$L(W, \mathcal{F}) = \overline{W}$, we can assume $\overline{W} = \overline{W'} = X$, and $W, W' \subset X$ are open affine. Let $U := W \cap W'$. It is enough to show that for any simple holonomic $\mathcal{H} \in \mathcal{H}ol(\mathcal{D}_X)$, $\mathcal{H} \cong j_{!*}\mathcal{H}|_U$. We have

$$\mathrm{Hom}(\mathcal{H}, j_*(\mathcal{H}|_U)) \cong \mathrm{Hom}(\mathcal{H}|_U, \mathcal{H}|_U) \cong \mathrm{Hom}(j_!(\mathcal{H}|_U), \mathcal{H})$$

Thus, $\mathcal{H}$ is a simple submodule of $j_*(\mathcal{H}|_U)$. But the only simple submodule of $j_*(\mathcal{H}|_U)$ is $j_{!*}(\mathcal{H}|_U)$. Thus $\mathcal{H} \cong j_{!*}(\mathcal{H}|_U)$.

□

Example 17.3. Let a connected algebraic group G act on X with finitely many orbits. Then any strongly G-equivariant coherent $\mathcal{D}_X$-module is holonomic (since its singular support is a union of cotangent bundles to orbits). By the theorem, every simple strongly G-equivariant $\mathcal{D}_X$-module is the !*-extension of a simple smooth G-equivariant $\mathcal{D}$-module on an orbit. A smooth simple strongly equivariant module on an orbit is in turn an irreducible representation of the component group of the stabilizer.

Here, by an equivariant G-module we mean that $g(dm) = g(d)g(m)$ where $m \in M$, and $d \in \mathcal{D}_X$. Then the Lie algebra $\mathfrak{g}$ of G acts on M in two ways: one obtained by deriving the action of G, and the other through the map $\mathfrak{g} \to \tau_X \to \mathcal{D}_X$. If the two actions coincide we call the module strongly equivariant.

Exercise 17.4. Let $\mathbb{K} = \mathbb{C}$, and $j : \mathbb{A}^1 \setminus \{0\} \hookrightarrow \mathbb{A}^1$. Compute $j_{!*}\mathcal{F}$ and $Cone(\varphi_{\mathcal{F}})$ if $\mathcal{F}$ is the $\mathcal{D}(\mathbb{A}^1 \setminus \{0\})$-module generated by the function f on $\mathbb{R}_{>0}$ where:

(i) $f(x) = x^\lambda$ for some $\lambda \in \mathbb{C}$
(ii) $f(x) = \log x$

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