INTERTWINING OPERATORS BETWEEN LINE BUNDLES ON GRASSMANNIANS

DMITRY GOUREVITCH AND SIDDHARTHA SAHI

Abstract. Let $G = GL(n, F)$ where F is a local field of arbitrary characteristic, and let π_1, π_2 be representations induced from characters of two maximal parabolic subgroups P_1, P_2. We explicitly determine the space $Hom_G(\pi_1, \pi_2)$ of intertwining operators and prove that it has dimension ≤ 1 in all cases.

1. Introduction

Let G be a reductive group over a local field F; then $C^\infty(G)$ is a $G \times G$-module with left and right actions given by $L_g f(x) = f(g^{-1}x)$, $R_g f(x) = f(xg)$. Let $P \subset G$ be a parabolic subgroup with modular function Δ_P and let χ be a character of P. The induced representation $I(P, \chi)$ is the right G-action on the space

$$C^\infty(G, P, \chi) := \left\{ f \in C^\infty(G) \mid L_{p^{-1}} f = \chi(p) \Delta_P^{1/2}(p) f \text{ for all } p \in P \right\}, \tag{1}$$

whose elements may also be regarded as smooth sections of a line bundle on G/P.

We are primarily interested in the group $G = G_n := GL(n, F)$ and its parabolic subgroups $P = P_{p_1,p_2}$, with $p_1 + p_2 = n$, consisting of matrices $x \in G_n$ of the form

$$x = \begin{bmatrix} x_{11} & x_{12} \\ 0 & x_{22} \end{bmatrix}; x_{ij} \in Mat_{p_i \times p_j}. \tag{2}$$

In this case G/P is the Grassmannian of p_1-dimensional subspaces of F^n. The characters of P are of the form $\chi_1 \otimes \chi_2(x) = \chi_1(x_{11}) \chi_2(x_{22})$, where χ_i is a character of G_{p_i}. Following [BZ77] we write $\chi_1 \times \chi_2$ instead of $I(P, \chi_1 \otimes \chi_2)$.[1]

Let $\pi_1 = \chi_1 \times \chi_2$ and $\pi_2 = \chi_3 \times \chi_4$ be two such representations, where each χ_i is character of G_{p_i} with $p_1 + p_2 = p_3 + p_4 = n$. Our main result is an explicit determination of the space $Hom_{G_n}(\pi_1, \pi_2)$ of intertwining operators, or *intertwiners* for short; in particular we prove that it has dimension at most 1.

It turns out that all intertwiners were previously known. The list includes such examples as the Radon transform and cosine transform, which are of considerable geometric interest; indeed these transforms were first constructed and studied in a geometric context, their intertwining properties being only recognized much later [GGR84, A10]. One can further supplement the list with two simple examples, scalar operators for $\pi_1 = \pi_2$, and certain rank 1 operators obtained as a composition of two rank 1 Radon transforms. Finally, for the middle Grassmannian over archmidean fields ($F = \mathbb{R}$ or $\mathbb{C}$), one also has discrete families of intertwiners given by certain Capelli-type *differential* operators. Our main contribution, in addition to the multiplicity 1 statement, is to show that there are no other intertwiners.

We fix some notation to describe our main results succinctly. For $z \in F$ let $\nu(z)$ denote the positive scalar by which the additive Haar measure on F transforms under multiplication by z. We will also regard ν as a character of G_n defined by $\nu(g) := \nu(\det g)$, and we note that the modular function of $P = P_{p_1,p_2}$ is

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[1] If p_1 (resp. p_2) $= 0$ then $P = G$ and $\chi_1 \times \chi_2 = \chi_2$ (resp. χ_1).

$\Delta_P = \nu^{p_2} \otimes \nu^{-p_1}$. For integers $i \le j$ we write $[i, j)$ for the character $\nu^{\frac{i+j}{2}}$ of G_{j-i}. If $\pi = \chi_1 \times \chi_2$ then we write $\tilde{\pi} = \chi_2 \times \chi_1$. Finally we write $\pi_1 \dashrightarrow \pi_2$ to mean that there exists a non-zero intertwining operator from π_1 to π_2.

Proposition 1.1. *For any $\pi = \chi_1 \times \chi_2$ we have $\pi \dashrightarrow \pi$ and $\pi \dashrightarrow \tilde{\pi}$.*

Proposition 1.2. *Fix an integer $k > 0$ and for each integer $0 \le i < k$ define $\alpha_i = [0, i) \times [i, k)$; then for all integers $0 \le j \ne i < k$ we have*

$$\widetilde{\alpha_j} \dashrightarrow \alpha_i \ .$$

Proposition 1.3. *Fix integers $0 < i < j < k$ and define $\beta = [0, j) \times [i, k)$, $\gamma = [0, k) \times [i, j)$, then we have*

$$\gamma \dashrightarrow \beta, \tilde{\gamma} \dashrightarrow \beta, \tilde{\beta} \dashrightarrow \gamma, \tilde{\beta} \dashrightarrow \tilde{\gamma}.$$

Proposition 1.4. *Fix an integer $k > 0$ and let 1, δ, ς denote the trivial,* det*, and sgn* (det) *characters of $GL_k(\mathbb{R})$; then for all integers $i > 0$ we have*

$$1 \times \delta^i \varsigma \dashrightarrow \delta^i \times \varsigma.$$

Proposition 1.5. *Fix an integer $k > 0$ and let 1, δ, $\bar{\delta}$ denote the trivial,* det*, and* $\overline{\det}$ *characters of $GL_k(\mathbb{C})$, then for all integers $i > 0$ and all integers j we have*

$$1 \times \delta^i \bar{\delta}^j \dashrightarrow \delta^i \times \bar{\delta}^j, 1 \times \bar{\delta}^i \delta^j \dashrightarrow \bar{\delta}^i \times \delta^j \ .$$

We can get additional instances of $\pi_1 \dashrightarrow \pi_2$ by considering central twists. To formulate this precisely we introduce the following notation.

Notation 1.6. *Given non-negative integers $p_1 + p_2 = p_3 + p_4 = n$ and characters χ_i of G_{p_i} we write $\mathfrak{X} = (\chi_1, \chi_2, \chi_3, \chi_4)$,*

$$H(\mathfrak{X}) = Hom_{G_n}(\chi_1 \times \chi_2, \chi_3 \times \chi_4)$$

We define the *central twist* of $\mathfrak{X}$ by a character ψ of $F^\times$ to be

$$\psi\mathfrak{X} = (\psi\chi_1, \dots, \psi\chi_4) \text{ with } (\psi\chi_i)(g) = \psi(\det g)\chi_i(g).$$

It is easy to see that for all ψ we have a natural isomorphism $H(\mathfrak{X}) \approx H(\psi\mathfrak{X})$.

Definition 1.7. *We refer to the $\mathfrak{X}$ obtained by central twists from Propositions 1.1 (resp. 1.2,1.3) (resp. 1.4, 1.5) as* standard *(resp.* mixed) *(resp.* exceptional*).*

Our main result is as follows.

Theorem 1.8. $\dim H(\mathfrak{X}) \le 1$ *with equality iff $\mathfrak{X}$ is standard, mixed, or exceptional.*

The intertwiners in the standard case are either scalar operators or Knapp-Stein operators (cosine transforms). In the mixed case the intertwiners are rank 1 operators in Proposition 1.2, and Radon transforms in Proposition 1.3. The intertwiners in the exceptional case in Propositions 1.4 and 1.5 are given by explicit differential operators. In the real case they factor through a Speh representation and in the complex case either their domain or their range are irreducible.

In §2 we give some preliminaries on induced representations of reductive groups. In §3 we introduce a key tool: the Bernstein-Zelevinsky theory of derivatives. This tool is specific for G_n, but works uniformly over all fields. In §4 we construct the intertwining operators. In §5 we finish the proof of Theorem 1.8. The proof is carried out by induction, using the theory of derivatives and results on finite-dimensional subquotients.

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2. Preliminaries

2.1. **Degenerate principal series.** Let G be a reductive group over an arbitrary local field F. In this section we discuss some basic properties of the induced representation $I(P,\chi)$ on $C^{\infty}(G,P,\chi)$ as in (1). For detailed proofs we refer the reader to [BW00, Wall88] and to other standard texts on representation theory.

Let $\mathcal{E}'(G)$ denote the set of compactly supported distributions on G, regarded as a left and right G-module as usual via the pairing $\langle\cdot,\cdot\rangle : \mathcal{E}'(G)\times C^{\infty}(G)\to\mathbb{C}$.

Lemma 2.1. *Let $\varepsilon\in\mathcal{E}'(G)$ denote evaluation at $1\in G$, then we have*

$$\left\langle R_{p^{-1}}\varepsilon, f\right\rangle = \chi(p)\,\Delta_P^{1/2}(p)\,\langle\varepsilon, f\rangle \text{ for all } p\in P, f\in C^{\infty}(G,P,\chi)$$

Proof. Indeed both sides are equal to $f(p)$. □

Lemma 2.2. *The representations $I(P,\chi)$ and $I\left(P,\chi^{-1}\right)$ are contragredient.*

Proof. This is proved in [Wall88, V.5.2.4]. □

Let $\bar{P}$ denote the parabolic subgroup opposite to P. Then the characters of P and $\bar{P}$ can be identified with those of the common Levi subgroup $L=P\cap\bar{P}$.

Proposition 2.3. *There is a nonzero intertwining operator $I(P,\chi)\to I\left(\bar{P},\chi\right)$.*

Proof. Let $\chi_s=\chi\Delta_P^s$. By [KnSt80, Th. 6.6] and [Wald03, Th. IV.1.1] there is a family of intertwining operators $A(s)=I(P,\chi_s)\to I\left(\bar{P},\chi_s\right)$ depending meromorphically on the complex parameter s. Taking the principal part at $s=0$, i.e. choosing an integer k such that $s^kA(s)$ has a finite non-zero limit as $s\to 0$, we get the result. □

2.2. **Finite dimensional representations.** Let (ϕ,V) be an irreducible finite dimensional representation of a reductive group G. We are interested in the possibility of realizing ϕ as a submodule or quotient of some $I(P,\chi)$, which we denote by $\phi\hookrightarrow I(P,\chi)$ and $I(P,\chi)\twoheadrightarrow\phi$ respectively. We start with two simple results.

Lemma 2.4. *We have $\dim V^{P,\chi}\leq 1$ for all χ, with equality for at most one χ.*

Proof. This is obvious if $\dim V=1$, while $\dim V>1$ only occurs in the archimedean case, where the result follows from highest weight theory. □

Let (ϕ^*,V^*) be the contragredient representation of (ϕ,V).

Lemma 2.5. *We have $\phi\hookrightarrow I(P,\chi)$ (resp. $I(P,\chi)\twoheadrightarrow\phi$) iff $(V^*)^{P,\chi^{-1}\Delta_P^{-1/2}}$ (resp. $V^{P,\chi\Delta_P^{-1/2}}$) is nonzero. For a given P, there is at most one such χ in each case.*

Proof. If $\phi\hookrightarrow I(P,\chi)$ then by Lemma 2.1 the restriction $\varepsilon|_V$ gives an element in $(V^*)^{P,\chi^{-1}\Delta_P^{-1/2}}$, easily seen to be nonzero. Conversely the matrix coefficient with respect to such an element provides an imbedding $\phi\hookrightarrow I(P,\chi)$. Next by Lemma 2.2, we see that

$$I\left(P,\chi^{-1}\right)\twoheadrightarrow\phi^* \iff \phi\hookrightarrow I(P,\chi) \iff (V^*)^{P,\chi^{-1}\Delta_P^{-1/2}}\neq 0.$$

Replacing ϕ by ϕ^* and χ by χ^{-1}, we deduce $I(P,\chi)\twoheadrightarrow\phi \iff V^{P,\chi\Delta_P^{-1/2}}\neq 0$.

The second part of the Lemma follows from Lemma 2.4. □

Proposition 2.6. *If $\phi\hookrightarrow I(P,\chi)$ (resp. $I(P,\chi)\twoheadrightarrow\phi$) then ϕ is the unique irreducible submodule (resp. quotient) of $I(P,\chi)$.*

Proof. By Lemma 2.2 it suffices to deal with that case $\phi \hookrightarrow I(P,\chi)$. If P is minimal, then the result follows from the Langlands classification ([La89],[Sil78]), once we note that by Lemma 2.5 and the dominance of the highest weight vector, ϕ is a Langlands submodule of $I(P,\chi)$. Otherwise choose a minimal parabolic $P_0 \subset P$. Then we have

$$\phi \hookrightarrow I(P,\chi) \subset I(P_0,\chi_0) \text{ with } \chi_0 = \Delta_{P_0}^{-1/2}\left(\chi\Delta_P^{1/2}\right)|_{P_0}.$$

But ϕ is the unique submodule of $I(P_0,\chi_0)$, hence also of $I(P,\chi)$. □

Fix a minimal parabolic subgroup $P_0 \subset G$ and let $\mathcal{M}$ be the set of pairs (P,χ) such that P is a *maximal* parabolic containing P_0 and χ is a character of P.

Lemma 2.7. *If* $\dim V > 1$ *then* $\phi \hookrightarrow I(P,\chi)$ *(resp.* $I(P,\chi) \twoheadrightarrow \phi$*) for at most one* $(P,\chi) \in \mathcal{M}$.

Proof. By Lemma 2.5 it suffices to show that if $(V^*)^{P_1,\chi_1}, (V^*)^{P_2,\chi_2} \neq 0$ for $(P_i,\chi_i) \in \mathcal{M}$ then $P_1 = P_2$. By the Lemmas 2.4 and 2.5 we conclude that χ_1, χ_2 have the same restriction χ_0 (say) to P_0, and that

$$(V^*)^{P_1,\chi_1} = (V^*)^{P_0,\chi_0} = (V^*)^{P_2,\chi_2}$$

If P_1, P_2 were *different* maximal parabolic subgroups then they would generate G, and the one-dimensional space $(V^*)^{P_0,\chi_0}$ would be G-invariant, contradicting the assumption that V, and hence V^*, is irreducible of dimension > 1. □

2.3. **Intertwining differential operators.** In this subsection we suppose that F is an archimedean field. Let G be a real reductive group, let $P = LN$ be a parabolic subgroup and denote the opposite nilradical by $\bar{N}$. We denote the Lie algebras of $G, \bar{N}$ etc. by $\mathfrak{g}, \bar{\mathfrak{n}}$ etc. and their enveloping algebras by $\mathcal{U}(\mathfrak{g})$, $\mathcal{U}(\bar{\mathfrak{n}})$ etc. The left and right G-actions on $C^\infty(G)$ give rise to vector fields L_X, R_X for $X \in \mathfrak{g}$, and more generally to differential operators L_u,R_u for $u \in \mathcal{U}(\mathfrak{g})$.

We are interested in triples (u,χ,η) where $u \in \mathcal{U}(\bar{\mathfrak{n}})$ and χ, η are characters of P such that L_u maps the space $C^\infty(G,P,\chi)$ to $C^\infty(G,P,\eta)$. Since left and right actions commute, such an L_u is automatically an intertwining differential operator between the induced representations $I(P,\chi)$ and $I(P,\eta)$, and we will refer to (u,χ,η) as an *intertwining triple.*

Proposition 2.8. *Suppose (a)* $\mathfrak{n}$ *is abelian, (b)* u *transforms by the character* $\chi\eta^{-1}$ *under the adjoint action of* L*, and (c) the product* $\chi\eta$ *extends to a character of* G*; then* (u,χ,η) *is an intertwining triple*

Proof. If $\eta = \chi^{-1}$ then this is proved in [KV77, Proposition 2.3], and the same proof works for the general case. □

Remark 2.9. *In the context of Proposition 2.8, since* $\bar{\mathfrak{n}}$ *is abelian, we may identify* $\mathcal{U}(\bar{\mathfrak{n}})$ *with the symmetric algebra* $\mathcal{S}(\bar{\mathfrak{n}})$*. Furthermore, we may identify* $\bar{\mathfrak{n}}$ *with* $\mathfrak{n}^*$ *and thus regard* $u \in \mathcal{S}(\bar{\mathfrak{n}})$ *as polynomial function on* $\mathfrak{n}$*.*

3. Derivatives

If χ is a character of G_p with $p > 0$, we write χ' for its restriction to G_{p-1} and χ^+ for its extension to G_{p+1} (*i.e.* the unique character such that $(\chi^+)' = \chi$). If $\mathfrak{X} = (\chi_1,\chi_2,\chi_3,\chi_4)$ with all $p_i > 0$ then we define

$$\mathfrak{X}' = (\chi_1',\chi_2',\chi_3',\chi_4'), \mathfrak{X}^+ = (\chi_1^+,\chi_2^+,\chi_3^+,\chi_4^+).$$

Lemma 3.1. *If all* $p_i > 1$ *and* $\mathfrak{X}'$ *is standard, mixed or exceptional then so is* $\mathfrak{X}$*.*

Proof. Since all $p_i > 1$ we have $\mathfrak{X} = (\mathfrak{X}')^+$. The result is obvious if $\mathfrak{X}'$ is standard or exceptional since $\delta'_p = \delta_{p-1}$ etc. For the mixed case we note that if $i < j$ then

$$\chi = [i, j) \implies \nu^{1/2}\chi^+ = \nu^{\frac{i+j+1}{2}} = [i, j+1).$$

Now writing $\sim$ to denote equality up to a (common) central twist, we see that

$$\mathfrak{X}' \sim ([i_1, j_1), \cdots, [i_4, j_4)) \implies \mathfrak{X} \sim ([i_1, j_1 + 1), \cdots, [i_4, j_4 + 1))$$

It follows easily that if $\mathfrak{X}'$ is mixed then, up to a twist, $\mathfrak{X}$ is as in Lemma 1.3. □

We will prove the main result (Theorem 1.8) by induction on n, using ideas from [BZ77, AGS]. We refer the reader to those papers for the notion of *depth* for an admissible representation of G_n, and for the definition of the functor Φ which maps admissible representations of G_n of depth ≤ 2 to admissible representations of G_{n-2}. In [BZ77] this functor is denoted Φ^-.

Proposition 3.2. *(*[BZ77, AGS]*)*

1. *Φ is an exact functor and $\Phi(\chi_i \times \chi_j) = \chi'_i \times \chi'_j$ if $p_i, p_j > 1$.*
2. *Every subquotient of $\chi_i \times \chi_j$ has depth ≤ 2*
3. *If π has depth 1 then π is finite dimensional and $\Phi(\pi) = 0$.*
4. *If π has depth 2 then $\Phi(\pi) \neq 0$*

Let $H(\mathfrak{X})$ be as in Notation 1.6 and we let $H_0(\mathfrak{X}) \subset H(\mathfrak{X})$ denote the subspace of finite rank operators.

Corollary 3.3. *If $H_0(\mathfrak{X}) = 0$ then Φ defines an imbedding $H(\mathfrak{X}) \hookrightarrow H(\mathfrak{X}')$.*

4. Construction of intertwining operators

We now prove Propositions 1.1 – 1.5. As before we write $\pi_1 \dashrightarrow \pi_2$ if there exists a non-zero intertwining operator from π_1 to π_2.

Proof of Proposition 1.1. The identity operator gives $\pi \dashrightarrow \pi$. Next we write

$$P = P_{p_1,p_2}, \chi = \chi_1 \otimes \chi_2, \tilde{P} = P_{p_2,p_1}, \tilde{\chi} = \chi_2 \otimes \chi_1.$$

Then we have $\pi = I(P, \chi)$ and $\tilde{\pi} = I\left(\tilde{P}, \tilde{\chi}\right) \approx I\left(\bar{P}, \chi\right)$ since $\left(\tilde{P}, \tilde{\chi}\right)$ and $\left(\bar{P}, \chi\right)$ are G-conjugate. Now we get $\pi \dashrightarrow \tilde{\pi}$ from Proposition 2.3. □

Proof of Proposition 1.2. It follows from Lemma 2.5 that $\phi \hookrightarrow \alpha_i$ and $\widetilde{\alpha_j} \twoheadrightarrow \phi$, where ϕ is the character $\nu^{k/2}$ of G_k. Thus we get a non-zero map $\widetilde{\alpha_j} \to \phi \to \alpha_i$. □

Proof of Proposition 1.3. By Proposition 1.2 and induction by stages we get maps

$$\gamma \to [0, j) \times [j, k) \times [i, j) \to \beta, \, \tilde{\gamma} \to [i, j) \times [0, i) \times [i, k) \to \beta,$$
$$\tilde{\beta} \to [i, k) \times [0, i) \times [i, j) \to \gamma, \, \tilde{\beta} \to [0, j) \times [j, k) \times [i, j) \to \tilde{\gamma}.$$

To see that the composite maps are non-zero, we note that each map is non-zero on the one-dimensional space of vectors fixed by the maximal compact subgroup. □

Proof of Proposition 1.4. Let $G = GL_{2k}(\mathbb{R})$ and $P = P_{k,k}$ then $\mathfrak{n} \approx Mat_{k\times k}(\mathbb{R})$ is abelian. Let $u \in U(\bar{\mathfrak{n}})$ correspond, as in Remark 2.9, to the polynomial function $\det^i$ on $\mathfrak{n}$, and set

$$\chi = 1 \otimes \delta^i \varsigma, \eta = \delta^i \otimes \varsigma.$$

Then u transforms by the character $\chi\eta^{-1} = \delta^{-i} \otimes \delta^i$ under the adjoint action of $L = G_k \times G_k$, and the product $\chi\eta = \delta^i \otimes \delta^i$ extends to the character δ^i of $G = G_{2k}$. Thus (u, χ, η) is an intertwining triple by Proposition 2.8, and the result follows. □

Proof of Proposition 1.5. This is proved similarly, using the polynomial functions $\det^i$ and $\overline{\det}^i$ on $\mathfrak{n} \approx Mat_{k\times k}(\mathbb{C})$. □

Remark 4.1. *In Proposition 1.4 the maps factor through the Speh representation (see* [SaSt90]*). In Proposition 1.5, either the source or the target of the map are irreducible (see* [HL99]*).*

5. Proof of Theorem 1.8

Let $H_0(\mathfrak{X}) \subset H(\mathfrak{X})$ denote the subspace of maps of finite rank. If $H_0(\mathfrak{X}) \neq 0$, then there is a finite-dimensional representation ϕ that is a quotient of $\chi_1 \times \chi_2$ and a submodule of $\chi_3 \times \chi_4$. We will indicate this by writing $\phi \vdash \mathfrak{X}$.

Proposition 5.1. *We have* $\dim H(\mathfrak{X}) \leq 1$*.*

Proof. First suppose $H_0(\mathfrak{X}) \neq 0$, and let $\phi \vdash \mathfrak{X}$ be as above. By Proposition 2.6 ϕ is the unique irreducible quotient of $\chi_1 \times \chi_2$ and the unique irreducible submodule of $\chi_3 \times \chi_4$. It follows that any map in $H(\mathfrak{X})$ factors through ϕ and hence $\dim H(\mathfrak{X}) = 1$.

If $H_0(\mathfrak{X}) = 0$ then by Corollary 3.3 we get an imbedding $H(\mathfrak{X}) \hookrightarrow H(\mathfrak{X}')$. The result now follows by induction on $n = p_1 + p_2 = p_3 + p_4$, with the initial cases $n = 0$ and $n = 1$ being trivial. □

Lemma 5.2. *If* $H_0(\mathfrak{X}) \neq 0$ *then* $\mathfrak{X}$ *is standard or mixed.*

Proof. Let $\phi \vdash \mathfrak{X}$ be as above. If $\dim \phi = 1$, then Lemma 2.5 implies that up to a central twist by ϕ we have

$$\chi_1 \times \chi_2 = [j, n) \times [0, j) \text{ and } \chi_3 \times \chi_4 = [0, i) \times [i, n) \text{ for some } 0 \leq i, j < n.$$

Thus if $i = j$ then $\mathfrak{X}$ is standard, otherwise $\mathfrak{X}$ is mixed as in Lemma 1.2.

If $\dim \phi > 1$ then F is archimedean and by Lemma 2.5 ϕ is a submodule of $\chi_2 \times \chi_1$. By Lemma 2.7 ϕ is a submodule of a unique degenerate principal series. Thus $\chi_3 = \chi_2$ and $\chi_4 = \chi_1$ and hence $\mathfrak{X}$ is standard. □

Before proving the next result we make some simple observations.

Lemma 5.3. *Let* $\mathfrak{X} = (\chi_1, \chi_2, \chi_3, \chi_4)$ *with* $H(\mathfrak{X}) \neq 0$*, and write* $\chi_i = \psi_i \circ \delta_{p_i}$ *then*

$$\psi_1(z)^{p_1} \psi_2(z)^{p_2} = \psi_3(z)^{p_3} \psi_4(z)^{p_4} \text{ for all } z \in F^\times. \tag{3}$$

Proof. It follows from the definition of induction that the central element $zI_n \in G_n$ acts on $\chi_1 \times \chi_2$ and $\chi_3 \times \chi_4$ by the scalars $\psi_1(z)^{p_1} \psi_2(z)^{p_2}$ and $\psi_3(z)^{p_3} \psi_4(z)^{p_4}$. If $Hom_{G_n}(\chi_1 \times \chi_2, \chi_3 \times \chi_4) \neq 0$ then these scalars must be the same. □

Corollary 5.4. *Let* $\mathfrak{X} = (\chi_1, \chi_2, \chi_3, \chi_4)$ *with* $H(\mathfrak{X}) \neq 0$*.*

(i) If $p_1 = 1$ *then* χ_1 *is uniquely determined by* χ_2, χ_3, χ_4*.*

(ii) If $p_1 = p_3 = 1$ *and* $\chi_2 = \chi_4$*, then* $\chi_1 = \chi_3$*.*

Proof. For case (i) we note that $\psi_1(z) = \psi_2(z)^{-p_2} \psi_3(z)^{p_3} \psi_4(z)^{p_4}$ by (3). In case (ii) we have $p_2 = p_4 = n - 1$ and $\psi_2 = \psi_4$, hence by(3) we get $\psi_1 = \psi_3$. □

Proposition 5.5. *If* $H(\mathfrak{X}) \neq 0$ *then* $\mathfrak{X}$ *is standard, mixed, or exceptional.*

Proof. We proceed by induction on $n = p_1 + p_2 = p_3 + p_4$. The case $n = 1$ is obvious and the case $n = 2$ follows from standard facts about principal series for GL_2 (see e.g. [Wall88, §§5.6,5.7] for the archimedean case). Thus from now on we may assume that $n \geq 3$ and that the result is true for $n - 2$. By the previous lemma we may also assume that $H_0(\mathfrak{X}) = 0$. In particular we may assume that each $p_i > 0$ and by induction that $\mathfrak{X}'$ is standard, mixed, or exceptional.

Let I be the set of indices i such that $p_i = 1$. Since $n \geq 3$, I can contain at most one index from each of the sets $\{1, 2\}$ and $\{3, 4\}$. If $I = \emptyset$ then the result follows from Lemma 3.1.

Suppose $|I|=1$. If $I=\{1\}$ then up to a central twist we have

$$H(\mathfrak{X}')=Hom\left([0,n-2),[0,p_3-1)\times[p_3-1,n-2)\right),$$

and hence we get $\mathfrak{X}\sim(\chi_1,[0,n-1),[0,p_3),[p_3-1,n-1))$. By Corollary 5.4 we must have $\chi_1=[p_3-1,p_3)$, and hence $\mathfrak{X}$ is mixed. The proof is similar if $I=\{2\},\{3\}$ or $\{4\}$.

Suppose $|I|=2$. If $I=\{1,3\}$ then up to a central twist we get

$$H(\mathfrak{X}')=Hom\left([0,n-2),[0,n-2)\right).$$

Thus we have $\mathfrak{X}\sim(\chi_1,[0,n-1),\chi_3,[0,n-1))$. By Corollary 5.4 we get $\chi_1=\chi_3$ and hence $\mathfrak{X}$ is standard. The proof is similar in the other cases with $|I|=2$. □

Proof of Theorem 1.8. This follows from Propositions 1.1 – 1.5, 5.1 and 5.5. □

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Dmitry Gourevitch, Faculty of Mathematics and Computer Science, Weizmann Institute of Science, POB 26, Rehovot 76100, Israel
E-mail address: dimagur@weizmann.ac.il
URL: http://www.wisdom.weizmann.ac.il/~dimagur

Siddhartha Sahi, Department of Mathematics, Rutgers University, Hill Center - Busch Campus, 110 Frelinghuysen Road Piscataway, NJ 08854-8019, USA
E-mail address: sahi@math.rugers.edu