INVARIANT FUNCTIONALS ON SPEH REPRESENTATIONS

DMITRY GOUREVITCH, SIDDHARTHA SAHI, AND EITAN SAYAG

ABSTRACT. We study $\mathrm{Sp}_{2n}(\mathbb{R})$-invariant functionals on the spaces of smooth vectors in Speh representations of $\mathrm{GL}_{2n}(\mathbb{R})$.

For even n we give expressions for such invariant functionals using an explicit realization of the space of smooth vectors in the Speh representations. Furthermore, we show that the functional we construct is, up to a constant, the unique functional on the Speh representation which is invariant under the Siegel parabolic subgroup of $\mathrm{Sp}_{2n}(\mathbb{R})$. For odd n we show that the Speh representations do not admit an invariant functional with respect to the subgroup U_n of $\mathrm{Sp}_{2n}(\mathbb{R})$ consisting of unitary matrices.

Our construction, combined with the argument in [GOSS12], gives a purely local and explicit construction of Klyachko models for all unitary representations of $\mathrm{GL}_n(\mathbb{R})$.

1. INTRODUCTION

In recent years, there has been considerable interest in periods of automorphic forms in relation to the Langlands program and equidistribution problems ([SV, Ven10]). The study of periods admits a local counterpart: the study of invariant linear functionals and the concomitant notion of *distinction* of a representation π of a reductive group G with respect to a subgroup $H \subset G$. We recall that a representation π is called **distinguished** with respect to a subgroup $H \subset G$ if the **multiplicity space** $\mathrm{Hom}_H(\pi^\infty, \mathbb{C})$ of H-invariant continuous functionals on the space π^∞ of smooth vectors of π is non-zero. In many interesting cases the pair (G, H) is a Gelfand pair, which means that the dimension of the multiplicity space is at most one for any irreducible admissible representation π of G. This allows one to connect the global period integral to local linear functionals. Motivated by the work of Jacquet-Rallis [JR92] and Heumos-Rallis [HR90], the third author together with O. Offen classified in [OS07, OS08a, OS08b, OS09] those unitary representations of $\mathrm{GL}_{2n}(F)$ that are distinguished with respect to the subgroup $\mathrm{Sp}_{2n}(F)$, in the case that F is a non-archimedean local field. The case of archimedean F was treated subsequently in [GOSS12, AOS12]. We remark that the pair $\mathrm{Sp}_{2n}(F) \subset \mathrm{GL}_{2n}(F)$ is a Gelfand pair (see [OS08b, AS12, Say]).

The classification of $\mathrm{Sp}_{2n}(\mathbb{R})$-distinguished unitary representations of $\mathrm{GL}_{2n}(\mathbb{R})$ involves the family of unitary representations discovered by B. Speh ([Sp83]). We recall that these unitary representations and their generalizations to $\mathrm{GL}_n(F)$, where F is a local field, play a central role in the Tadic-Vogan classification of the unitary dual of $\mathrm{GL}_n(F)$. To describe this classification we use the Bernstein-Zelevinsky notation $\pi_1 \times \pi_2$ for (normalized) parabolic induction from $\mathrm{GL}_{n_1}(F) \times \mathrm{GL}_{n_2}(F)$ to $\mathrm{GL}_{n_1+n_2}(F)$. For a discrete series representation σ of $\mathrm{GL}_r(F)$ we denote by $U(\sigma, n)$ the corresponding Speh representation of $\mathrm{GL}_{nr}(F)$, and by

$$\pi(\sigma, n, \alpha) := U(\sigma, n)|\cdot|^\alpha \times U(\sigma, n)|\cdot|^{-\alpha}, \ 0 < \alpha < \frac{1}{2}$$

the corresponding Speh complementary series representation.

Then any irreducible unitary representation of $\mathrm{GL}_m(F)$ can be written in the form

$$\pi = \pi_1 \times \cdots \times \pi_k, \tag{1}$$

where each π_i is either a $U(\sigma_i, n_i)$ or a $\pi(\sigma_i, n_i, \alpha_i)$, and such an expression is unique up to reordering of the π_i (see [Tad86, Vog86]). The answer to the distinction is summarized in the next theorem, which in the archimedean case is a combination of [GOSS12, Theorem A] and [AOS12, Theorem 1.1].

Theorem. *If π is an irreducible unitary representation of $\mathrm{GL}_{2n}(F)$ as in (1), then π is $\mathrm{Sp}_{2n}(F)$-distinguished iff all the n_i are even.*

One of the key steps in the proof is to show that the generalized Speh representations $U(\sigma, n)$ with even n are distinguished by the symplectic group. The proof of this result in [OS07] and [GOSS12] is based on a global argument involving periods of residues of automorphic Eisenstein series.

Recall that for archimedean F we have $r \leq 2$, and if $F = \mathbb{C}$ then $r = 1$. If $r = 1$ then $U(\sigma, n)$ is a character of $\mathrm{GL}_n(F)$, and $\pi(\sigma, n, \alpha)$ is a Stein complementary series representation of $\mathrm{GL}_{2n}(F)$. We denote by D_m the discrete series representations of $\mathrm{GL}_2(\mathbb{R})$ and by δ_m the corresponding Speh representations of $\mathrm{GL}_{2n}(\mathbb{R})$. In [SaSt90] the Speh representations δ_m of $\mathrm{GL}_{2n}(\mathbb{R})$ have been constructed explicitly as natural Hilbert spaces of distributions on matrix space. The paper [SaSt90] also describes and uses a construction of the Speh representations as quotients of degenerate principal series representations induced from characters of the (n, n) standard parabolic subgroup (see §2.2 below).

In the present paper we use the explicit constructions of [SaSt90] and give a direct proof that the spaces of $\mathrm{Sp}_{2n}(\mathbb{R})$-invariant functionals on the Speh representations of $\mathrm{GL}_{2n}(\mathbb{R})$ are zero if n is odd and one-dimensional if n is even. We also analyze functionals invariant with respect to subgroups of $\mathrm{Sp}_{2n}(\mathbb{R})$.

To describe our result we need some further notation. Let $G := G_{2n}$ denote the group $\mathrm{GL}_{2n}(\mathbb{R})$. Let ω_{2n} be the standard symplectic form on $\mathbb{R}^{2n}$. More explicitly ω_{2n} is given by $\begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}$ and let $H := H_{2n} = Sp_{2n}(\mathbb{R}) \subset G_{2n}$ denote the stabilizer of this form. Let

$$P := \left\{ \begin{pmatrix} g & X \\ 0 & (g^t)^{-1} \end{pmatrix} \mid g \in \mathrm{GL}_n(\mathbb{R}), X \in \mathrm{Mat}_{n \times n}(\mathbb{R}), X = X^t \right\} \subset H$$

denote the Siegel parabolic subgroup. Let $\mathrm{U}_n \subset H_{2n} \subset G_{2n}$ be the unitary group.

In this paper we prove the following result.

Theorem A.   *(i) If n is even then*

$$\mathrm{Hom}_H(\delta_m^\infty, \mathbb{C}) = \mathrm{Hom}_P(\delta_m^\infty, \mathbb{C}) \simeq \mathbb{C}$$

(ii) If n is odd then

$$\mathrm{Hom}_H(\delta_m^\infty, \mathbb{C}) = \mathrm{Hom}_{\mathrm{U}_n}(\delta_m^\infty, \mathbb{C}) = \{0\}.$$

It is known that the restriction of δ_m to $\mathrm{SL}_{2n}(\mathbb{R})$ decomposes as a direct sum of two irreducible components $\delta_m^\pm$. It follows from Theorem A that exactly one of them admits an H-invariant functional. In Lemma 4.2 we determine that δ_m^+ does.

It is easy to see that if n is odd and m is even then there are no functionals on δ_m^∞ invariant with respect to $-\mathrm{Id} \in P \cap \mathrm{U}_n$, and thus neither P-invariant nor U_n-invariant functionals exist (see Remark 6.1).

Remark. *Although the pair (G, P) is* **not** *a Gelfand pair for simple geometric reasons, we show that the Speh representation δ_m still admits at most one P-invariant functional (at least for even n). The reason we suspected this result to hold is that, as shown in [SaSt90], Speh representations stay irreducible when restricted to a standard maximal parabolic subgroup $Q \subset G$ satisfying $Q \cap H = P$. It is possible that (Q, P) is a generalized Gelfand pair, i.e. the space of P-invariant functionals on the space of smooth vectors of any irreducible unitary representation of Q is at most one dimensional. However, this statement would still not imply our uniqueness result, since the space of G-smooth vectors of δ_m could a priori afford more continuous functionals.*

1.1. **Related results.** The present work was motivated by our previous results on Klyachko models for unitary representations of $\mathrm{GL}_n(\mathbb{R})$. For any n, any even $k \leq n$ and any field F, [Kly84] defines a subgroup Kl_k of $\mathrm{GL}_n(F)$ and a generic character ψ_k of Kl_k. In particular, Kl_0 is the group of upper unitriangular matrices and $Kl_n = \mathrm{Sp}_n(F)$ (if n is even). It is shown in [Kly84, IS91, HZ00] for finite fields F and in [HR90, OS07, OS08a, OS08b, OS09, GOSS12, AOS12] for local fields F that for any irreducible unitary representation π of $\mathrm{GL}_n(F)$ there exists a non-zero (Kl_k, ψ_k)-equivariant functional on π^∞ for exactly one k. The uniqueness of such functional is known only over non-archimedean fields (see [OS08b]).

The proof of existence of k for $F = \mathbb{R}$, given in [GOSS12], is achieved by reduction to the statement that certain representations of $G = \mathrm{GL}_{2n}(\mathbb{R})$ are $H = \mathrm{Sp}_{2n}(\mathbb{R})$-distinguished. This statement is further reduced, using the Vogan classification of the unitary dual, to an existence statement of H-invariant functionals on the Speh representations (for even n). Finally, the existence statement is proved using a global (adelic) argument. In the present paper we give an explicit local construction of such a functional. Together with [GOSS12] this gives a proof of existence of Klyachko models which uses only the representation theory of $\mathrm{GL}_n(\mathbb{R})$ (and the theory of distributions).

The study of invariant functionals in this paper, and more broadly the study of multiplicity spaces belongs to the long and classical tradition of branching laws (see e.g. [GW09, Chapter 8]). In the context of symmetric pairs and more generally in the context of spherical spaces, the basic result is that these multiplicity spaces are finite dimensional ([KO13], cf. [KrSch]). Granted this qualitative result, one turns to the question of precisely determining the dimension. We note that in many interesting cases these spaces are at most one-dimensional (see e.g. [vD86, AG09, AGRS10, SZ12]). This multiplicity one phenomenon has important consequences in number theory ([Gross91]).

In some situations there are precise conjectures as to the dimensions of these multiplicity spaces (see e.g. [GGP12, Wald12]) but in general these dimensions are hard to determine, even in the context of symmetric pairs. Another important task, motivated in part by the theory of automorphic forms, is to construct a basis for these multiplicity spaces. Recently, there has been a considerable interest in these aspects of the theory under the title of *symmetry breaking*. The general theory of branching laws attempts the description of symmetry breaking operators occurring in the general context of restrictions of representations as in [KoSp14, KoSp15]. In particular, it is interesting to compare our main result with [KoSp15, Chapter 14].

Another related result is the exact branching of the representations $\delta_m^{\pm}$ of $\mathrm{SL}(4, \mathbb{R})$ to $\mathrm{Sp}(4, \mathbb{R})$ as analyzed in [OrSp08]. It is shown there that the decomposition of δ_m^{-} is discrete and multiplicity free, while the decomposition of δ_m^{+} is continuous.

1.2. **Structure of the proof.** We use the realization of δ_m^{∞} as the image of a certain intertwining differential operator $\square^m : \pi_{-m}^{\infty} \to \pi_m^{\infty}$, where π_{-m}^{∞} and π_m^{∞} are degenerate principal series representations induced from certain characters of a fixed (n, n)-parabolic subgroup $\overline{Q} \subset G$ (see §2.2).

The study of the even case is divided into two parts. In §3 we first use the realization of δ_m^{∞} as a quotient of the degenerate principal series π_{-m}^{∞} to lift a linear P-invariant functional on δ_m^{∞} to an equivariant distribution on G. More precisely, we study $P \times \overline{Q}$ equivariant distributions on G. The technical heart is Corollary 3.3, which shows that such distributions do not vanish on the open cell $N\overline{Q}$. This is based on the techniques of [AGS08], classical invariant theory and a careful analysis of the double cosets $P \backslash G / \overline{Q}$, which is postponed to §5. Then we analyze the space of distributions on the open cell $N\overline{Q}$ by identifying it with the space of distributions on N with a certain equivariance property. Identifying N with its Lie algebra and using the Fourier transform we show that this space is at most one-dimensional for even n. This finishes the proof of Proposition 3.1 which states that there exists at most one P-invariant functional in the n even case.

In the second part (§4) we construct an H-invariant functional as an $H \times \overline{Q}$-equivariant distribution on G. For that we fix an explicit $H \times \overline{Q}$-equivariant non-negative polynomial p, consider the meromorphic family of distributions p^{λ} (cf. [Ber72]) and take the principal part of this family at $\lambda = (n - m)/2$, *i.e.* the lowest non-zero coefficient in the Laurent expansion. This distribution defines an H-invariant functional on π_m^{∞}. To show that the restriction of this functional to δ_m^{∞} is non-zero (Lemma 4.1) we use Corollary 3.3 along with another lemma from §3 on non-existence of equivariant distributions with certain support. The uniqueness of P-invariant functionals and the existence of H-invariant ones imply that the two spaces are equal. Our proof shows that the spaces of such functionals are equal and one-dimensional also for the (reducible) representations π_m^{∞} and π_{-m}^{∞}.

For odd n we prove that already a U_n-invariant functional does not exist (Corollary 6.4). We do that by analyzing the $\mathrm{O}_{2n}(\mathbb{R})$-types of δ_m described in [HL99, Sah95] and showing that none of those have a U_n-invariant vector.

To summarize, Theorem A follows from Proposition 3.1 on uniqueness of P-invariant functionals for even n, Lemma 4.1 on existence of H-invariant functionals for even n and Corollary 6.4 on non-existence of U_n-invariant functionals for odd n.

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2. Preliminaries

2.1. **Notation.** Recall the notation $G = G_{2n} = \mathrm{GL}_{2n}(\mathbb{R})$, and $H = H_{2n} = \mathrm{Sp}_{2n}(\mathbb{R}) \subset G$. Let

$$Q := \left\{ \begin{pmatrix} a & c \\ 0 & d \end{pmatrix} \in G \right\} \quad \overline{Q} := \left\{ \begin{pmatrix} a & 0 \\ b & d \end{pmatrix} \in G \right\} \quad N := \left\{ \begin{pmatrix} \mathrm{Id}_n & c \\ 0 & \mathrm{Id}_n \end{pmatrix} \in G \right\}.$$

Recall that P denotes $Q \cap H$ and let

$$M := \left\{ \begin{pmatrix} g & 0 \\ 0 & (g^t)^{-1} \end{pmatrix} \right\} \quad \text{and} \quad U := \left\{ \begin{pmatrix} \mathrm{Id}_n & B \\ 0 & \mathrm{Id}_n \end{pmatrix} \mid B = B^t \right\}$$

denote the Levi subgroup and the unipotent radical of P.

For $g \in \mathrm{Mat}_{i \times i}(\mathbb{R})$ we denote $|g| := |\det(g)|$ and $\mathrm{sgn}(g) := \mathrm{sgn}(\det(g))$.

For $q = \begin{pmatrix} A & 0 \\ B & D \end{pmatrix} \in \overline{Q}$ we denote $\gamma(q) := |A||D|^{-1}$ and $\varepsilon(q) := \mathrm{sgn}(D)$.

For any integer m let L_m denote the character of $\overline{Q}$ given by $L_m := \varepsilon^{m+1} \gamma^{-(n+m)/2}$. Let π_m^∞ denote the (unnormalized) induced representation $\mathrm{Ind}_{\overline{Q}}^G(L_m)$, with the topology of uniform convergence on $G/\overline{Q}$ together with all the derivatives. Considering N as an open subset of $G/\overline{Q}$, one can restrict smooth vectors of π_m^∞ to N. This restriction is an embedding since N is an open subset of $G/\overline{Q}$. We sometimes identify N and its Lie algebra $\mathfrak{n}$ with $\mathrm{Mat}_{n \times n}(\mathbb{R})$ by

$$\begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \mapsto X \quad \text{and} \quad \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} \mapsto X.$$

This enables us to define the Fourier transform on $\mathfrak{n}$. Denote by M_n^+ (respectively M_n^-) the subset of $\mathrm{Mat}_{n \times n}(\mathbb{R})$ consisting of matrices with nonnegative (resp. nonpositive) determinant. For $f \in \pi_m^\infty$ we denote its restriction to $\mathfrak{n}$ by $f|_\mathfrak{n}$. We denote the space of all smooth functions obtained in this way by $\pi_m^\infty|_\mathfrak{n}$.

2.2. **Sahi-Stein realization of the Speh representations.** For any $m \in \mathbb{Z}_{\geq 0}$ define

$$\widehat{H}_m := \{ f \in \mathcal{S}^*(\mathfrak{n}) \mid \widehat{f} \in L^2(\mathfrak{n}, |x|^{-m}dx) \} \quad \text{and} \quad \widehat{H}_m^\pm := \{ f \in \widehat{H}_m \mid \mathrm{Supp}\widehat{f} \subset M_n^\pm \},$$

where $\mathcal{S}^*(\mathfrak{n})$ denotes the space of tempered distributions $\mathfrak{n}$. The $\widehat{H}_m$ and $\widehat{H}_m^\pm$ are Hilbert spaces with the scalar product

$$\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle_{L^2(\mathfrak{n}, |x|^{-m}dx)}.$$

Define an action of Q on $\widehat{H}_m$ by

$$\delta_m(q)f(x) := L_m(q)f(a^{-1}(c + xd)), \quad \text{for } q = \begin{pmatrix} a & c \\ 0 & d \end{pmatrix},$$

or equivalently on the Fourier transform side by

$$\widehat{\delta_m(q)}f(\xi) = \exp(2\pi i \, \mathrm{Tr}(cd^{-1}\xi))L_{-m}^{-1}(q)\widehat{f}(d^{-1}\xi a).$$

Summarizing the main results of [SaSt90] we obtain

Theorem 2.1 ([SaSt90]). *Let $m \in \mathbb{Z}_{\geq 0}$. Then*

(i) *The action of Q extends to a unitary representation δ_m of G on $\widehat{H}_m$.*

(ii) *$(G, \delta_m, \widehat{H}_m)$ is isomorphic to the Speh representation of G.*

(iii) *There exists an epimorphism $\pi^\infty_{-m} \to \delta^\infty_m$ and an embedding $\delta^\infty_m \subset \pi^\infty_m$. The latter is defined by the inclusion $\delta^\infty_m \subset \pi^\infty_m|_\mathfrak{n}$.*

(iv) *The restriction of δ_m to $\mathrm{SL}(2n, \mathbb{R})$ is a direct sum of two irreducible representations $\delta^\pm_m$, realized on the subspaces $\widehat{H}^\pm_m$.*

Consider the determinant as a polynomial on $\mathfrak{n}$ and let $\square$ denote the corresponding differential operator.

Theorem 2.2. *The operator $\square^m$ defines a continuous $\mathrm{SL}(2n, \mathbb{R})$-equivariant map $\pi^\infty_{-m} \to \pi^\infty_m$ with image δ^∞_m.*

Proof. We will prove a stronger statement: the operator $\square^m$ defines a continuous G-equivariant map $\pi^\infty_{-m} \to \mathrm{sgn}\pi^\infty_m$ with image δ^∞_m, where $\mathrm{sgn}\pi^\infty_m$ denotes the twist of π^∞_m by the sign character of G.

By [KV77, Proposition 2.3] (see also [Boe85]), the operator $\square^m$ defines a continuous G-equivariant map $\pi^\infty_{-m} \to \mathrm{sgn}\pi^\infty_m$, which is non-zero by [SaSt90]. By [HL99, Theorems 3.4.2-3.4.4] (see also [Sah95]) π^∞_{-m} has unique composition series in the strong sense, meaning that any quotient of π^∞_{-m} has a unique irreducible subrepresentation, and all these irreducible subquotients are pairwise non-isomorphic. It is easy to see that π^∞_m is dual to π^∞_{-m} and thus their composition series are opposite. The composition series are described in [HL99] in terms of their K-types, where $K = \mathrm{O}(2n, \mathbb{R})$ is the maximal compact subgroup of G, and it is easy to see from this description that the set of K-types of the irreducible quotient of π^∞_{-m} is invariant under multiplication by sgn. By the result of Casselman and Wallach (see [Cas89] or [Wall92, Chapter 11]), the category of smooth admissible Fréchet representations of moderate growth is abelian and any morphism in it has closed image. Hence the image of any nonzero intertwining operator from π^∞_{-m} to $\mathrm{sgn}\pi^\infty_m$ is the unique irreducible quotient of π^∞_{-m}. Since δ^∞_m is an irreducible quotient of π^∞_{-m}, the image of $\square^m$ is δ^∞_m. $\square$

Remark 2.3. *One can deduce Theorem 2.2 also from [KS93], which computes the action of $\square^m$ on every K-type, where $K = \mathrm{O}(2n, \mathbb{R})$. From the formula in [KS93] and the description of the K-types of the composition series of π^∞_{-m} in [HL99, Sah95] one can see that $\square^m$ does not vanish precisely on the K-types of δ^∞_m.*

2.3. Invariant distributions. We will now recall some generalities on Schwartz functions and tempered distributions.

Definition 2.4. *For an affine algebraic manifold M we denote by $\mathcal{S}(M)$ the space of Schwartz functions on M, that is smooth functions f such that df is bounded for any differential operator d on M with algebraic coefficients. We endow this space with a Fréchet topology using the sequence of seminorms $\mathcal{N}_d(f) := \sup_{x \in M} |df(x)|$, where d is a differential operator on M with algebraic coefficients. Also, for an algebraic vector bundle E over M we denote by $\mathcal{S}(M, E)$ the space of Schwartz sections of E. We denote by $\mathcal{S}^*(M, E)$ the space of continuous linear functionals on $\mathcal{S}(M, E)$ and call its elements tempered distributional sections. For a closed subvariety $Z \subset M$ we denote by $\mathcal{S}^*_M(Z, E) \subset \mathcal{S}^*(M, E)$ the subspace of tempered distributional sections supported in Z. For the theory of Schwartz functions and distributions on general semi-algebraic manifolds we refer the reader to [AG08].*

Notation 2.5.
- *For a manifold M and closed submanifold $Z \subset M$ we denote by $N^M_Z := TM|_Z/TZ$ the normal bundle to Z in M and by $CN^M_Z \subset T^*M$ its dual bundle, i.e. the conormal bundle to Z in M.*
- *For a point $z \in Z$ we denote by $N^M_{Z,z}$ the normal space at z to Z in M and by $CN^M_{Z,z}$ the conormal space at z to Z in M.*
- *For a group K acting on a vector space V we denote by V^K the subspace of K-invariant vectors and by $V^{K,\chi}$ the subspace of vectors that change by the character χ.*
- *If K acts on a manifold M we denote by $\mathcal{S}^*(M)^{K,\chi}$ the space of distributions on M that change by the character χ under the action of K.*
- *For a real algebraic group K we denote by Δ_K its modular character.*

Theorem 2.6 ([AGS08, §B.2])**.** *Let a real algebraic group K act on a real algebraic manifold M. Let $Z \subset M$ be a Zariski closed subset. Let $Z = \bigcup_{i=1}^{l} Z_i$ be a K-invariant stratification of Z. Let χ be a character of K. Suppose that for any $k \in \mathbb{Z}_{\geq 0}$ and $1 \leq i \leq l$,*

$$\mathcal{S}^*(Z_i, \mathrm{Sym}^k(CN_{Z_i}^M))^{K,\chi} = \{0\}.$$

Then $\mathcal{S}_M^*(Z)^{K,\chi} = \{0\}$.

Theorem 2.7 (Frobenius descent, see [AG09, Appendix B])**.** *Let a real algebraic group K act on a real algebraic manifold M. Let Z be a real algebraic manifold with a transitive action of K. Let $\phi : M \to Z$ be a K-equivariant map. Let $z \in Z$ be a point and $M_z := \phi^{-1}(z)$ be its fiber. Let K_z be the stabilizer of z in K. Let E be a K-equivariant algebraic vector bundle over M.*

Then there exists a canonical isomorphism

$$\mathrm{Fr} : (\mathcal{S}^*(M_z, E|_{M_z}) \otimes \Delta_K|_{K_z} \cdot \Delta_{K_z}^{-1})^{K_z} \cong \mathcal{S}^*(M, E)^K.$$

From those two theorems we obtain the following corollary.

Corollary 2.8. *Let a real algebraic group K act on a real algebraic manifold M. Let $Z \subset M$ be a Zariski closed subset. Suppose that Z has a finite number of orbits: $Z = \bigcup_{i=1}^{l} K z_i$. Let χ be a character of K. Suppose that for any $1 \leq i \leq l$ we have*

$$\mathrm{Sym}^*(N_{Kz_i, z_i}^M)^{K_{z_i}, \chi \cdot \Delta_K|_{K_{z_i}} \cdot \Delta_{K_{z_i}}^{-1}} = \{0\},$$

where Sym^* denotes the symmetric algebra. Then $\mathcal{S}_M^*(Z)^{K,\chi} = \{0\}$.

Lemma 2.9. *Let K be a real algebraic group, and R be a (closed) algebraic subgroup. Consider the right action of R on K and suppose that K/R is compact. Let ξ be a character of R. Then we have a natural isomorphism of left K - representations*

$$(\mathrm{Ind}_R^K(\xi))^* \cong (C^\infty(K, \xi)^R)^* \cong \mathcal{S}^*(K, \xi \Delta_K|_R \Delta_R^{-1})^R \cong \mathcal{S}^*(K)^{(R, \xi \Delta_K|_R \Delta_R^{-1})}.$$

Proof. The first and the last isomorphisms are straightforward. Let us prove the one in the middle.

Let $\mathfrak{Ind}(\xi)$ be the bundle on K/R corresponding to ξ. Consider the surjective submersion $\pi : K \to K/R$. It defines an isomorphism $C^\infty(K, \xi)^R \cong C^\infty(K/R, \mathfrak{Ind}(\xi))$.

Since K/R is compact, we have $C^\infty(K/R, \mathfrak{Ind}(\xi))^* \cong \mathcal{S}^*(K/R, \mathfrak{Ind}(\xi))$. Consider the diagonal action of K on $K \times K/R$ and the projections p_1, p_2 of $K \times K/R$ on both coordinates. From Theorem 2.7 we obtain

$$\mathcal{S}^*(K/R, \mathfrak{Ind}(\xi)) \cong \mathcal{S}^*(K \times K/R, p_1^*(\xi))^K \cong \mathcal{S}^*(K, \xi \Delta_K|_R \Delta_R^{-1})^R.$$

$\square$

3. Uniqueness of P-invariant functionals

In this section we assume that n is even. The goal of this section is to prove the following proposition.

Proposition 3.1. *For any integer m we have*

$$\dim((\pi_m^\infty)^*)^P \leq 1.$$

Recall the character L_m of $\overline{Q}$ from §2.1 and note that $L_{-m}^{-1} = \varepsilon^{m+1} \gamma^{(n-m)/2}$. Since $\Delta_{\overline{Q}} = \gamma^{-n}$, we obtain from the definition of π_m^∞ and Lemma 2.9

$$(2) \qquad\qquad\qquad (\pi_m^\infty)^* \cong \mathcal{S}^*(G)^{\overline{Q}, L_{-m}^{-1}}$$

and thus in order to prove Proposition 3.1 we have to show that for even n

$$\dim \mathcal{S}^*(G)^{P \times \overline{Q}, 1 \times L_{-m}^{-1}} \leq 1.$$

We will need the following proposition, which we will prove in section 5.

Proposition 3.2. *Denote $K := P \times \overline{Q}$, and let $x \notin N\overline{Q}$. Then*

$$\mathrm{Sym}^*(N_{Px\overline{Q}, x}^G)^{K_x, L_{-m}^{-1} \cdot \Delta_K|_{K_x} \Delta_{K_x}^{-1}} = \{0\}.$$

From this proposition and Corollary 2.8 we obtain

Corollary 3.3.

$$\mathcal{S}_G^*(G - N\overline{Q})^{P \times \overline{Q}, 1 \times L_{-m}^{-1}} = \{0\}.$$

By this corollary it is enough to analyze $\mathcal{S}^*(N\overline{Q})^{P \times \overline{Q}, 1 \times L_{-m}^{-1}}$. Let S denote the space of symmetric $n \times n$ matrices, and A denote the space of anti-symmetric $n \times n$ matrices. Identify $M \cong \mathrm{GL}_n(\mathbb{R})$ and let it act on S and on A by $x \mapsto gxg^t$.

Lemma 3.4. *We have*

$$\mathcal{S}^*(N\overline{Q})^{P \times \overline{Q}, 1 \times L_{-m}^{-1}} \cong \mathcal{S}^*(A)^{\mathrm{GL}_n(\mathbb{R}), \det^{1-m}} \cong \mathcal{S}^*(A)^{\mathrm{GL}_n(\mathbb{R}), \mathrm{sgn}^{m+1} |\cdot|^{m-n}}$$

Proof. Identify $U \cong S$ and let it act on itself by translations. Then $N\overline{Q}$ is isomorphic as a $P \times \overline{Q}$-space to $A \times S \times \overline{Q}$, where $\overline{Q}$ acts on the third coordinate (by right translations), U acts on the second coordinate and M acts on the first and the second coordinates. Note that the action of $P \times \overline{Q}$ on $S \times \overline{Q}$ is transitive and that $\Delta_{\overline{Q}} = \gamma^{-n}$ and $\Delta_P \begin{pmatrix} g & 0 \\ 0 & (g^{-1})^t \end{pmatrix} = |g|^{n+1}$. The first isomorphism follows now from Frobenius descent.

The second isomorphism is given by Fourier transform on A defined using the trace form. $\qquad\square$

Let $O \subset A$ denote the open dense subset of non-degenerate matrices and Z denote its complement. The following lemma is a straightforward computation.

Lemma 3.5.

(i) *Every orbit of* $\mathrm{GL}_n(\mathbb{R})$ *in* Z *includes an element of the form* $x := \begin{pmatrix} 0_{k \times k} & 0 \\ 0 & \omega_{n-k} \end{pmatrix}$, *for some even k.*

(ii) $N_{\mathrm{GL}_n(\mathbb{R})x,x}^A \cong \left\{ \begin{pmatrix} 0_{k \times k} & b \\ 0 & 0 \end{pmatrix} \right\}$ *and* $\mathrm{GL}_n(\mathbb{R})_x = \left\{ \begin{pmatrix} a_{k \times k} & 0 \\ c & d \end{pmatrix} \text{ such that } d \in \mathrm{Sp}_{(n-k)} \right\}$.

(iii) $\Delta_{\mathrm{GL}_n(\mathbb{R})_x} = |\cdot|^{-(n-k)}$.

Corollary 3.6. *For any $x \in Z$ we have*

$$\mathrm{Sym}^*(N_{\mathrm{GL}_n(\mathbb{R})x,x}^A)^{\mathrm{GL}_n(\mathbb{R})_x, \mathrm{sgn}^{m+1} |\cdot|^{m-n} \cdot \Delta_{\mathrm{GL}_n(\mathbb{R})_x}^{-1}} = \{0\}.$$

Proof. From the previous lemma $\mathrm{sgn}^{m+1} |\cdot|^{m-n} \cdot \Delta_{\mathrm{GL}_n(\mathbb{R})_x}^{-1} = \mathrm{sgn}^{k+1} \det^{m-k} = \mathrm{sgn} \det^{m-k}$. This is not an algebraic character of $\mathrm{GL}_n(\mathbb{R})_x$ and thus there are no tensors that change under this character. $\qquad\square$

Corollary 3.7.

$$\dim \mathcal{S}^*(A)^{\mathrm{GL}_n(\mathbb{R}), \mathrm{sgn}^{m+1} |\cdot|^{m-n}} \leq 1.$$

Proof. By Corollary 3.6 and Corollary 2.8,

$$(3) \qquad\qquad \mathcal{S}_A^*(Z)^{\mathrm{GL}_n(\mathbb{R}), \mathrm{sgn}^{m+1} |\cdot|^{m-n}} = \{0\}.$$

Therefore, the restriction of equivariant distributions to O is an embedding. Now,

$$\dim \mathcal{S}^*(O)^{\mathrm{GL}_n(\mathbb{R}), \mathrm{sgn}^{m+1} |\cdot|^{m-n}} \leq 1,$$

since O is a single orbit. $\qquad\square$

Proposition 3.1 follows now from Corollary 3.7, Lemma 3.4, Corollary 3.3 and (2).

Remark 3.8. *Corollary 3.3 does not extend to the case of odd n. For example, in this case the closed $P \times \overline{Q}$-orbit $\overline{Q}$ does support an equivariant distribution.*

4. Construction of the H-invariant functional

Let n be even. In this section we construct an H-invariant functional ϕ on π_m^∞ for any $m \in \mathbb{Z}_{\geq 0}$ and show that its restriction to δ_m^∞ is non-zero. Define a polynomial p on $\mathrm{Mat}_{2n \times 2n}(\mathbb{R})$ by

$$(4) \qquad p \begin{pmatrix} A & B \\ C & D \end{pmatrix} := \det(D^t B - B^t D) = \mathrm{Pfaffian}^2(D^t B - B^t D).$$

Note that p is non-negative, H-invariant on the left and changes under the right multiplication by $\overline{Q}$ by the character $|\cdot|\gamma^{-1}$. Consider the meromorphic family of distributions on $\mathrm{Mat}_{2n \times 2n}(\mathbb{R})$ given by p^λ. This family is defined for $\mathrm{Re}\,\lambda > 0$ and by [Ber72] has a meromorphic continuation (as a family of distributions) to the entire complex plane. For $\mathrm{Re}\,\lambda > 0$, the restriction of this distribution to $G = \mathrm{GL}_{2n}(\mathbb{R})$ is a non-zero smooth function, and thus the restriction of the family to G is not identically zero. Define

$$(5) \qquad \eta_\lambda^m := (p^\lambda|_G)|\cdot|^{-\lambda}\varepsilon^{m+1}.$$

This is a tempered distribution, since $|\cdot|^\lambda$ is a smooth function on G of moderate growth. Note that

$$\eta_\lambda^m \in \mathcal{S}^*(G)^{(H \times \overline{Q}, 1 \times \varepsilon^{m+1}\gamma^\lambda)}.$$

Let $\alpha \in \mathcal{S}^*(G)$ be the principal part of this family at $\lambda = \frac{n-m}{2}$, i.e. the lowest non-zero coefficient in the Laurent expansion. By (2) α defines a non-zero H-invariant functional ϕ on π_m^∞. π_m^∞.

Lemma 4.1. $\phi|_{\delta_m^\infty} \neq 0$.

Proof. By Theorem 2.2 it is enough to show that $\square^m \phi \neq 0$. By Corollary 3.3, $\alpha|_{N\overline{Q}} \neq 0$. It is enough to show that $(\square^m \alpha)|_{N\overline{Q}} \neq 0$. As in §3, let $A \subset N$ denote the subspace of anti-symmetric matrices and $O \subset A$ the open subset of non-degenerate matrices. Note that $\alpha|_{N\overline{Q}} \neq 0$ is $P \times \overline{Q}$-equivariant and let $\beta \in \mathcal{S}^*(A)^{\mathrm{GL}_n(\mathbb{R}),\det^{1-m}}$ be the distribution on A corresponding to α by the Frobenius descent (see Lemma 3.4). Note that $\mathcal{F}(\square^m \beta)$ is $\mathcal{F}(\beta)$ multiplied by a polynomial. Thus it is enough to show that $\mathcal{F}(\beta)$ has full support, i.e. $\mathcal{F}(\beta)|_O \neq 0$. This follows from the equivariance properties of $\mathcal{F}(\beta)$ by (3). $\square$

This argument in fact proves slightly more.

Lemma 4.2. $\phi|_{(\delta_m^+)^\infty} \neq 0$.

Proof. If g is a Schwartz function on $M_n^+ \subset N$ then its Fourier transform $\widehat{g}$ defines a vector in $(\delta_m^+)^\infty$ by Theorem 2.1. Thus it is enough to find such a g for which $\zeta(\widehat{g}) \neq 0$, where ζ denotes the P-invariant distribution on N corresponding to α.

Let f be a compactly supported smooth function on O such that $\beta(\mathcal{F}(f)) \neq 0$. Since the determinant is positive on O, there exists a compact neighborhood Z of zero in the space S of symmetric n by n matrices such that $\mathrm{Supp}(f) + Z \subset M_n^+$. Let h be a smooth function on S which is supported on Z and s.t. $h(0) = 1$. Let $g := f \boxtimes h$ be the function on N defined by $g(X + Y) := f(X)h(Y)$ where $X \in A$ and $Y \in S$. Let $\mathcal{F}_S$ denote the Fourier transform on S. Then we have

$$\zeta(\widehat{g}) = \zeta(\mathcal{F}(f) \boxtimes \mathcal{F}_S(h)) = \beta(\mathcal{F}(f)) \neq 0.$$

$\square$

Remark 4.3. *(i) For odd n, the polynomial p is identically zero, since the matrix $D^t B - B^t D$ is an anti-symmetric matrix of size n.*

(ii) The polynomial p defines the open orbit of H on $G/\overline{Q}$. In general, one can show that if a linear complex algebraic group $\mathbf{K}$ acts with finitely many orbits on a complex affine algebraic manifold $\mathbf{M}$, both defined over $\mathbb{R}$, $\mathbf{W}$ is a basic open subset of M defined by a $\mathbf{K}$-equivariant polynomial p with real coefficients, χ is a character of the group of real points K of $\mathbf{K}$ and there exists a non-zero (K, χ)-equivariant tempered distribution ξ on W then there exists a non-zero (K, χ)-equivariant tempered distribution on M. Here, W and M denote the real points of $\mathbf{W}$ and $\mathbf{M}$.

To prove that consider the analytic family of distributions $|p|^\lambda \xi$ on W. For $\mathrm{Re}\,\lambda$ big enough, it can be extended to a family η_λ on M. By [Ber72] the family η_λ has a meromorphic continuation to the entire complex plane. Note that the distributions in this family are equivariant with a character

that depends analytically on λ. Thus taking the principal part at $\lambda = 0$ we obtain a non-zero (K, χ)-equivariant tempered distribution on M.

Note that since this construction involves taking principal part, the obtained distribution is not necessary an extension of the original ξ. This can already be seen in the case when $\mathbf{M} = \mathbb{C}$ is the affine line, $\mathbf{W}$ is the complement to 0 and $\mathbf{K}$ is the multiplicative group $\mathbb{C}^\times$.

5. Proof of Proposition 3.2

We start from the description of the double cosets $P \setminus G / \overline{Q}$. Let r_1, r_2, s, t be non-negative integers such that $r_1 + r_2 + 2s + 2t = n$. We will view $2n \times 2n$ matrices as 10×10 block matrices in the following way. First of all, we view them as 2×2 block matrices with each block of size $n \times n$. Now, we divide each block to 5×5 blocks of sizes $r_1, r_2, s, s, 2t$ in correspondence. Denote by σ_{16} the permutation matrix that permutes blocks 1 and 6, by σ_{39} the permutation matrix that permutes blocks 3 and 9, and by $\tau_{5,10}$ the matrix which has $\begin{pmatrix} \mathrm{Id}_{2t} & \omega_{2t} \\ 0 & \mathrm{Id}_{2t} \end{pmatrix}$ in blocks 5 and 10 and is equal to the identity matrix in other blocks. Recall the notation $\omega_{2t} := \begin{pmatrix} 0 & \mathrm{Id}_t \\ -\mathrm{Id}_t & 0 \end{pmatrix}$. Denote

$$(6) \qquad x_{r_1, r_2, s, t} := \sigma_{16} \sigma_{39} \tau_{5,10}.$$

Lemma 5.1. *Each double coset in $P \setminus \mathrm{GL}_{2n}(\mathbb{R}) / \overline{Q}$ includes a unique element of the form $x_{r_1, r_2, s, t}$. The orbits in $N\overline{Q}$ correspond to $r_1 = s = 0$.*

Proof. Consider the Lagrangian subspaces $L := \mathrm{Span}\{e_1, \dots e_n\} \subset \mathbb{R}^{2n}$ and $L' := \mathrm{Span}\{e_{n+1}, \dots e_{2n}\} \subset \mathbb{R}^{2n}$. Note that Q preserves L and $\overline{Q}$ preserves L'. Identify $G/\overline{Q}$ with the Grassmannian of n-dimensional subspaces of $\mathbb{R}^{2n}$ by $g \mapsto gL'$. To an n-dimensional subspace $W \subset \mathbb{R}^{2n}$ we associate the following invariants:

$$r_1 := \dim L \cap W \cap W^\perp, \ r_2 := \dim W^\perp \cap W - r_1, \ s := \dim L \cap W - r_1, \ t := (n - r_1 - r_2)/2 - s.$$

Note that $n - r_1 - r_2$ is even since it is the rank of $\omega|_W$. Note also that the identity $(L \cap W \cap W^\perp)^\perp = L + W + W^\perp$ implies $n \geq r_1 + r_2 + 2s$. Clearly, $W \in NL'$ if and only if $r_1 = s = 0$.

Note the equality of vectors

$$(7) \qquad (v_1, 0, v_2, 0, \omega_{2t} u \,|\, 0, w_2, w_1, 0, u)^t = x_{r_1, r_2, s, t}(0, 0, 0, 0, 0 \,|\, v_1, w_2, w_1, v_2, u)^t.$$

It is enough to show that W can be transformed, using the action of P, to a space of vectors of the form (7).

Let us first show that W can be transformed to a space of vectors of the form

$$(8) \qquad (v, Aw + Bv \,|\, Cw, w, Dw)^t, \qquad \text{where } \mathrm{size}(v) + \mathrm{size}(w) = n \text{ and } A \text{ is a square matrix.}$$

There exists a set S of n coordinates such that the projection of W on the space of vectors that have zero coordinates from S is an isomorphism. Suppose that S has k of the coordinates $1 \dots n$, and thus $n - k$ of the coordinates $n + 1, \dots 2n$. Note that acting by M we can perform any permutation of the first n coordinates followed the same permutation on the last n coordinates. Using such permutations we can transform S to the set $\{n - k + 1, \dots, n, n + 1, \dots n + l, n + k + l + 1, \dots, 2n\}$ for some $l \leq n - k$. Then W will have the form (8).

Let us now rewrite (8) in more detailed form, using four blocks of the same sizes y_i in the first n coordinates and last n coordinates:

$$(v_1, v_2, A_{11}w_1 + A_{12}w_2 + B_{11}v_1 + B_{12}v_2, A_{21}w_1 + A_{22}w_2 + B_{21}v_1 + B_{22}v_2 \,|\, C_1w_1 + C_2w_2, w_1, w_2, D_1w_1 + D_2w_2)^t$$

Denote the first four blocks by e_i and the last by f_i. For any $i, j \in \{1, 2, 3, 4\}$ with $i \neq j$, $M = \mathrm{GL}_n(\mathbb{R})$ allows us to do the following operations:

$$(1)_i \quad e_i \mapsto g e_i, \quad f_i \mapsto (g^t)^{-1} f_i, \quad \text{where } g \in \mathrm{GL}_{y_i}(\mathbb{R})$$

$$(2)_{ij} \quad e_i \mapsto e_i + a e_j, \quad f_j \mapsto f_j - a^t f_i, \quad \text{where } a \in \mathrm{Mat}_{y_i \times y_j}(\mathbb{R}).$$

Similarly, U allows us to do two more operations:

$$(3)_{ij} \quad e_i \mapsto e_i + b f_j, \quad e_j \mapsto e_j + b^t f_i, \quad \text{where } b \in \mathrm{Mat}_{y_i \times y_j}(\mathbb{R})$$

$$(4)_i \quad e_i \mapsto e_i + (c + c^t)f_i, \quad \text{where } c \in \text{Mat}_{y_i \times y_i}(\mathbb{R}).$$

Using $(2)_{31}$ and $(2)_{41}$, and redefining C and D we get $B = 0$. Using $(2)_{21}$ and $(2)_{34}$, and redefining A we get $C = 0$ and $D = 0$.

Using $(3)_{32}$ and $(3)_{42}$ and $(3)_{43}$ we can arrange $A_{11} = A_{21} = A_{22} = 0$. Using $(3)_{33}$ we make A_{12} anti-symmetric. Now, using $(1)_3$ we can replace A_{12} by $gA_{12}g^t$ and thus we can bring it to the form $A_{12} = \begin{pmatrix} 0 & 0 \\ 0 & \omega_{2t} \end{pmatrix}$.

$\square$

Lemma 5.2 (See §5.1 below). *Let* $K := P \times \overline{Q}$ *and* $x := x_{r_1,r_2,s,t}$. *Then*
(i) *If* $s > 0$ *then*
$$\text{Sym}^*(N^G_{Px\overline{Q},x}))^{K_x, L_{-m}^{-1} \cdot \Delta_K |_{K_x} \Delta_{K_x}^{-1}} = \{0\}.$$

(ii) *If* $s = 0$ *then*
$$\text{Sym}^*(N^G_{Px\overline{Q},x}))^{K_x, L_{-m}^{-1} \cdot \Delta_K |_{K_x} \Delta_{K_x}^{-1}} \cong \text{Sym}^*(\mathfrak{gl}_{r_1})^{\text{GL}_{r_1}, |\cdot|^{-m-r_1} \text{sgn}^{m+1}} \otimes \text{Sym}^*(o_{r_2})^{\text{GL}_{r_2}, \det^{2t-m+1}}$$

where o_{r_2} *denotes the space of antisymmetric matrices and* GL_{r_1} *and* GL_{r_2} *act by* $a \mapsto gag^t$.

Lemma 5.3. *Let* $k, l \in \mathbb{Z}_{\geq 0}$, $r \in \mathbb{Z}_{>0}$.
(i) *If* $k \neq l \pmod 2$ *then*
$$\text{Sym}^*(\mathfrak{gl}_r)^{\text{GL}_r, |\cdot|^k \text{sgn}^l} = \{0\}.$$

(ii) *If* $k \neq 0$ *and* r *is odd then*
$$\text{Sym}^*(o_r)^{\text{GL}_r, \det^k} = \{0\}.$$

Proof.
(i)The only algebraic characters of GL_r are powers of the determinant.
(ii) The stabilizer in GL_r of every matrix in o_r has an element with determinant different from 1. $\square$

Proof of Proposition 3.2. By Lemma 5.1 it is enough to show that for $x = x_{r_1,r_2,s,t}$ with $r_1 + s > 0$ we have
$$\text{Sym}^*(N^G_{Px\overline{Q},x}))^{K_x, L_{-m}^{-1} \cdot \Delta_K |_{K_x} \Delta_{K_x}^{-1}} = \{0\}.$$

If $s > 0$ this follows from Lemma 5.2(i). Otherwise $r_1 > 0$ and, by Lemma 5.2(i), we have to show that

$$(9) \qquad \text{Sym}^*(\mathfrak{gl}_{r_1})^{\text{GL}_{r_1}, |\det|^{-m-r_1} \text{sgn}(\det)^{m+1}} \otimes \text{Sym}^*(o_{r_2})^{\text{GL}_{r_2}, \det^{2t-m+1}} = \{0\}$$

Note that since n is even, r_1 and r_2 are of the same parity. If they are even then (9) follows from Lemma 5.3(i), and otherwise from Lemma 5.3(ii). $\square$

5.1. **Proof of Lemma 5.2.** Let $x = x_{r_1,r_2,s,t}$ be as in the lemma. We need to compute the space $N^G_{x,Px\overline{Q}}$, the stabilizer K_x and its modular function. In order to do that we compute the conjugates of P and its Lie algebra $\mathfrak{p}$ under x.

Lemma 5.4. *Let* $q := \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathfrak{q}$. *Then* $x^{-1}qx = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, *where*

$$A = \begin{pmatrix} d_{11} & 0 & d_{14} & 0 & 0 \\ b_{21} & a_{22} & b_{24} & a_{24} & a_{25} \\ d_{41} & 0 & d_{44} & 0 & 0 \\ b_{41} & a_{42} & b_{44} & a_{44} & a_{45} \\ b_{51} - \omega_{2t}d_{51} & a_{52} & b_{54} - \omega_{2t}d_{54} & a_{54} & a_{55} \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & d_{12} & d_{13} & 0 & d_{15} \\ a_{21} & b_{22} & b_{23} & a_{23} & b_{25} + a_{25}\omega_{2t} \\ 0 & d_{42} & d_{43} & 0 & d_{45} \\ a_{41} & b_{42} & b_{43} & a_{43} & b_{45} + a_{45}\omega_{2t} \\ a_{51} & b_{52} - \omega_{2t}d_{52} & b_{53} - \omega_{2t}d_{53} & a_{53} & b_{55} + a_{55}\omega_{2t} - \omega_{2t}d_{55} \end{pmatrix}$$

$$C = \begin{pmatrix} b_{11} & a_{12} & b_{14} & a_{14} & a_{15} \\ d_{21} & 0 & d_{24} & 0 & 0 \\ d_{31} & 0 & d_{34} & 0 & 0 \\ b_{31} & a_{32} & b_{34} & a_{34} & a_{35} \\ d_{51} & 0 & d_{54} & 0 & 0 \end{pmatrix} \quad D = \begin{pmatrix} a_{11} & b_{12} & b_{13} & a_{13} & b_{15} + a_{15}\omega_{2t} \\ 0 & d_{22} & d_{23} & 0 & d_{25} \\ 0 & d_{32} & d_{33} & 0 & d_{35} \\ a_{31} & b_{32} & b_{33} & a_{33} & b_{35} + a_{35}\omega_{2t} \\ 0 & d_{52} & d_{53} & 0 & d_{55} \end{pmatrix}.$$

This lemma is a straightforward computation.

We can identify $T_x G \cong \mathfrak{gl}_{2n}$. Under this identification $T_x P x \overline{Q} \cong x^{-1}\mathfrak{p}x + \overline{\mathfrak{q}}$ and

$$N^G_{x, Px\overline{Q}} \cong \mathfrak{gl}_{2n}/(x^{-1}\mathfrak{p}x + \overline{\mathfrak{q}}) \cong \mathfrak{n}/(\mathfrak{n} \cap (x^{-1}\mathfrak{p}x + \overline{\mathfrak{q}})).$$

From the previous lemma we obtain

Corollary 5.5. *Recall the identification* $n \cong \mathrm{Mat}_{n \times n}(\mathbb{R})$ *and let* $V \subset \mathfrak{n}$ *denote the subspace consisting of matrices of the form*

$$\begin{pmatrix} n_{11} & n_{12} & 0 & n_{14} & n_{15} \\ n_{12}^t & n_{22} & 0 & 0 & 0 \\ n_{31} & 0 & 0 & n_{34} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ n_{15}^t & 0 & 0 & 0 & 0 \end{pmatrix},$$

such that $n_{22} = -n_{22}^t$.

Then V *projects isomorphically onto* $\mathfrak{n}/(\mathfrak{n} \cap (x^{-1}\mathfrak{p}x + \overline{\mathfrak{q}}))$.

Now let us analyze the stabilizer K_x. From Lemma 5.4 we obtain

Corollary 5.6.

(i) *The Lie algebra* $\mathfrak{p} \cap x\overline{\mathfrak{q}}x^{-1}$ *consists of matrices* $\begin{pmatrix} A & B \\ 0 & -A^t \end{pmatrix}$ *such that*

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ 0 & A_{22} & 0 & 0 & 0 \\ 0 & A_{32} & A_{33} & 0 & -\omega_{2t}B_{35} \\ 0 & A_{42} & 0 & A_{44} & \omega_{2t}B_{45} \\ 0 & A_{52} & 0 & 0 & A_{55} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} & B_{13} & B_{14} & B_{15} \\ B_{12}^t & 0 & 0 & 0 & 0 \\ B_{13}^t & 0 & B_{33} & 0 & B_{35} \\ B_{14}^t & 0 & 0 & B_{44} & B_{45} \\ B_{15}^t & 0 & B_{35}^t & B_{45}^t & 0 \end{pmatrix},$$

$$A_{55} \in \mathfrak{sp}(2t), \ B_{11} = B_{11}^t, \ B_{33} = B_{33}^t, \ B_{44} = B_{44}^t.$$

(ii) *Let* $p = \begin{pmatrix} A & B \\ 0 & (A^t)^{-1} \end{pmatrix} \in P$. *Let* $k = (p, x^{-1}px) \in K_x$. *The modular function of* K_x *is given by*

$$\Delta_{K_x}(k) = |A_{11}|^{2n-r_1+1}|A_{22}|^{-n+r_1+r_2}|A_{33}|^{n-r_1-s+1}|A_{44}|^{n-r_1-s+1}.$$

(iii) *Let* $q = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \in \overline{Q} \cap x^{-1}Px$. *Let* $k = (xqx^{-1}, q) \in K_x$. *Then* k *acts on* V *by*

$$k \cdot n = pr_V(AnD^{-1}),$$

where $pr_V : \mathfrak{n} \to V$ *denotes the projection.*

Corollary 5.7. *Denote*

$$\chi := L_{-m}^{-1} \cdot \Delta_K|_{K_x}\Delta_{K_x}^{-1} = \varepsilon^{m+1}\gamma^{(n-m)/2} \cdot \Delta_K|_{K_x}\Delta_{K_x}^{-1}.$$

Let

$$q = \mathrm{diag}(a, b, c, (c^t)^{-1}, \mathrm{Id}, (a^t)^{-1}, (b^t)^{-1}, d, (d^t)^{-1}, \mathrm{Id}).$$

Let $k := (xqx^{-1}, q) \in K_x$. *Then*

$$\chi(k) = (\mathrm{sgn}(a)\,\mathrm{sgn}(b)\,\mathrm{sgn}(c)\,\mathrm{sgn}(d))^{m+1}|a|^{-m-r_1}|b|^{2s+2t-m+1}|c|^{-r_1-s}|d|^{-r_1-s}.$$

Proof.

$$\gamma(q) = |a|^2 |b|^2 \quad \text{and} \quad \Delta_{\overline{Q}}(q) = |a|^{-2n} |b|^{-2n}$$

$$xqx^{-1} = \text{diag}((a^t)^{-1}, b, (d^t)^{-1}, (c^t)^{-1}, \text{Id}, a, (b^t)^{-1}, d, c, \text{Id})$$

$$\Delta_K(k) = |a|^{-3n-1} |b|^{-n+1} |c|^{-n-1} |d|^{-n-1}$$

$$\Delta_{K_x}(k) = |a|^{-2n+r_1-1} |b|^{-n+r_1+r_2} |c|^{-n+r_1+s-1} |d|^{-n+r_1+s-1}$$

□

Now we are ready to prove Lemma 5.2.

Proof of Lemma 5.2. If $s > 0$ then $\text{Sym}^*(V)^{K_x,\chi} = 0$, since tensors cannot have negative homogeneity degrees. Otherwise, V involves only 3 blocks - the ones numbered $1, 2$ and 5.

Let $p \in \text{Sym}^*(V)^{K_x,\chi}$. Identify K_x with $x^{-1}Px \cap \overline{Q}$ using the projection on the second coordinate.

Consider the action of the block A_{21}. It can map any non-zero vector in the block n_{11} to any vector in the block n_{12}. This action does not change any element in any other block of V (it does effect n_{22}, but not its anti-symmetric part). Also, the character χ does not depend on A_{21}. Therefore p does not depend on the variables in the block n_{12}.

In the same way, using the action of A_{52}, we can show that p does not depend on the variables in the block n_{15}. Therefore, p depends only on n_{11} and n_{22}. Hence

$$\text{Sym}^*(V)^{K_x,\chi} \cong \text{Sym}^*(\mathfrak{gl}_{r_1})^{\text{GL}_{r_1},|\cdot|^{-m-r_1} \text{sgn}^{m+1}} \otimes \text{Sym}^*(o_{r_2})^{\text{GL}_{r_2},|\cdot|^{2t-m+1} \text{sgn}^{m+1}}.$$

□

6. Non-existence of an H-invariant functional for odd n

In this section we prove that if n is odd then there are no U_n-invariant functionals on the Speh representations and therefore there are no H-invariant functionals. We do that using K-type analysis. The maximal compact subgroup of G is $K := \text{O}_{2n}(\mathbb{R})$, and $\text{U}_n = K \cap H$ is a symmetric subgroup of K. We show that no K-type of δ_m has a U_n-invariant vector.

The root system of K is of type D_n, and we make the usual choice of positive roots

$$\{\varepsilon_i \pm \varepsilon_j : i < j\}$$

where ε_i is the i-th unit vector in $\mathbb{R}^n$. With this choice, the highest weights of K-modules are given by integer sequences $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$ such that

$$(10) \qquad\qquad\qquad \mu_1 \geq \cdots \geq \mu_{n-1} \geq \mu_n \geq 0.$$

If $\mu_n = 0$ then two K-types correspond to the sequence μ, they differ by the determinant character.

Remark 6.1. *From the definition of π_m^∞ we see that if n is odd and m is even then the central element $-\text{Id} \in G$ acts by scalar -1, and there are neither P-invariant nor U_n-invariant functionals on δ_m^∞.*

Let $m \geq 0$. Since δ_m^∞ is the irreducible quotient of π_{-m}^∞, the following theorem follows from [HL99, Theorems 3.4.2 - 3.4.4] (see also [Sah95]).

Theorem 6.2. *The K-types of $\pi_{\pm m}^\infty$ are given by sequences as in (10) with $\mu_i \equiv m + 1 \,(\text{mod } 2)$, while the K-types of the Speh representation δ_m satisfy the additional condition $\mu_n \geq m + 1$.*

Lemma 6.3. *If n is odd then no K-type $(\mu_1, \ldots, \mu_n)$ with $\mu_n \neq 0$ has U_n-invariant vectors.*

Proof. Denote $K^0 := \text{SO}_{2n}(\mathbb{R})$. By [Vog86, Proposition 5.17] any K-type with $\mu_n \neq 0$ decomposes into two K^0-types, one given by $(\mu_1, \ldots, \mu_n)$ and the other by $(\mu_1, \ldots, -\mu_n)$. Let us show that neither of them has U_n-invariant vectors.

Let $\mathfrak{a}$ be a maximal Cartan subspace of the symmetric pair $(\mathfrak{k}, \mathfrak{u}_n)$ and let $\mathfrak{t}$ be a maximal torus in the centralizer of $\mathfrak{a}$ in $\mathfrak{u}_n$. By the Cartan-Helgason theorem (see [Hel84, Chapter V, Corollary 4.2]) it suffices to show that the highest weights of the K^0-types mentioned above are not trivial when restricted to $\mathfrak{t}$.

Examining the Satake diagram in [Hel78, Chapter X, §F, Table VI, Case D III, odd r] we see that the Killing form identifies $\mathfrak{t}$ with the span of $\{\varepsilon_1 - \varepsilon_2, \varepsilon_3 - \varepsilon_4, \ldots, \varepsilon_{n-2} - \varepsilon_{n-1}, \varepsilon_n\}$. Thus the K^0-types that have U_n-invariant vectors are of the form $\mu_{2i-1} = \mu_{2i}$ for $1 \leq i \leq n/2$ and $\mu_n = 0$. $\qquad\square$

Corollary 6.4. *If n is odd then there are no U_n-invariant functionals on δ_m^∞.*

Proof. By Remark 6.1 we can assume that m is odd. Then by Lemma 6.3 and Theorem 6.2, no K-type of δ_m has a U_n-invariant vector. Therefore, the space of K-finite vectors, which decomposes to a direct sum of K-types, does not have a U_n-invariant functional. This space is dense in δ_m^∞, hence there are no U_n-invariant functionals on δ_m^∞ either. $\qquad\square$

Remark 6.5. *The pair (K^0, U_n), being a compact connected symmetric pair is a Gelfand pair. A similar argument to the one in the proof of Lemma 6.3 shows that (K, U_n) is also a Gelfand pair.*

REFERENCES

[AG08] A. Aizenbud, D. Gourevitch: *Schwartz functions on Nash Manifolds,* International Mathematics Research Notices, Vol. 2008, n.5, Article ID rnm155, 37 pages. DOI: 10.1093/imrn/rnm155. See also arXiv:0704.2891 [math.AG].

[AG09] A. Aizenbud, D. Gourevitch:*Generalized Harish-Chandra descent, Gelfand pairs and an Archimedean analog of Jacquet-Rallis' Theorem,* (with an appendix by E.Sayag), Duke Mathematical Journal, Volume 149, Number 3,509-567 (2009). See also arXiv: 0812.5063[math.RT].

[AGRS10] A. Aizenbud, D. Gourevitch, S. Rallis, G. Schiffmann: *Multiplicity One Theorems*, arXiv:0709.4215v1 [math.RT], Annals of Mathematics, Pages 1407-1434 from Volume 172 (2010), Issue 2.

[AGS08] A. Aizenbud, D. Gourevitch, E. Sayag: $(GL_{n+1}(F), GL_n(F))$ *is a Gelfand pair for any local field F*, postprint: arXiv:0709.1273v4[math.RT]. Originally published in: Compositio Mathematica, **144** , pp 1504-1524 (2008), doi:10.1112/S0010437X08003746.

[AOS12] A. Aizenbud, O. Offen, E. Sayag: *Disjoint pairs for* $GL(n, \mathbb{R})$ *and* $GL(n, \mathbb{C})$, Comptes Rendus Mathematique **350**, 9-11 (2012).

[AS12] A. Aizenbud, E. Sayag: *Invariant distributions on non-distinguished nilpotent orbits with application to the Gelfand property of (GL(2n,R),Sp(2n,R))*, Journal of Lie Theory **22** (2012), No. 1, 137–153. See also arXiv:0810.1853 [math.RT].

[Ber72] J. Bernstein: *Analytic continuation of generalized functions with respect to a parameter*, Functional Analysis and its Applications 6, No.4, 26-40 (1972).
differential operators; the study of

[Boe85] B. Boe: *Homomorphisms between generalized Verma modules,* Trans. A.M.S **288**, 791-799 (1985).

[BSS88] D. Barbasch, S. Sahi, B.Speh: *Degenerate series representations for GL(2n, R) and Fourier Analysis.* Symposia Mathematica, Vol. XXXI (1988, Rome), Sympos. Math., Vol. XXXI, Academic Press, London, 1990, 45-69.

[Cas89] W. Casselman: *Canonical extensions of Harish-Chandra modules to representations of G,* Can. J. Math., **XLI**, No. 3, 385-438 (1989).

[GGP12] W. Gan, H. Gross, D. Prasad: *Symplectic local root numbers, central critical L-values, and restriction problems in the representation theory of classical groups.* Asterisque **347**, 1-109 (2012).

[GW09] R. Goodman, N. R. Wallach: *Symmetry, Representations, and Invariants*, Graduate Texts in Mathematics **255**. Springer, Dordrecht xx+716 pp. (2009).

[GOSS12] D. Gourevitch, O. Offen, S. Sahi and E. Sayag: *Existence of Klyachko models for* $GL(n, \mathbb{R})$ *and* $GL(n, \mathbb{C})$, Journal of Functional Analysis **262**, Issue 8, pp 3465-3664 (2012). See also arXiv:1107.4621.

[Gross91] H. Gross: *Some applications of Gelfand pairs to number theory*, Bull. Amer. Math. Soc. (N.S.) Volume 24, Number 2 (1991), 277-301.

[Hel78] S. Helgason: *Differential geometry, Lie groups, and symmetric spaces,* Pure and Applied Mathematics, **80**. Academic Press, Inc., New York-London, xv+628 pp (1978).

[Hel84] S. Helgason: *Groups and geometric analysis,invariant differential operators, and spherical functions,* Pure and Applied Mathematics, **113**. Academic Press, Inc., Orlando, FL, xix+654 pp (1984).

[HL99] R. Howe, S. T. Lee: *Degenerate Principal Series Representations of* $GL_n(\mathbb{C})$ *and* $GL_n(\mathbb{R})$, Journal of Functional Analysis **166**, n. 2, 244-309 (1999).

[HR90] M. J. Heumos, S. Rallis: Symplectic-Whittaker models for GL_n. *Pacific J. Math.*, 146(2):247–279, 1990.

[HZ00] R. B. Howlett, C. Zworestine: On Klyachko's model for the representations of finite general linear groups. In *Representations and quantizations (Shanghai, 1998)*, pages 229–245. China High. Educ. Press, Beijing, 2000.

[IS91] N. F. J. Inglis, J. Saxl: An explicit model for the complex representations of the finite general linear groups. *Arch. Math. (Basel)*, 57(5):424–431 (1991).

[JR92] H. Jacquet and S. Rallis: Symplectic periods *J. Reine Angew. Math.*, **423** 175-197 (1992).

[Kly84] A. A. Klyachko: Models for complex representations of groups GL(n, q). *Mat. Sb. (N.S.)*, 48(2)(3):365–378 (1984).

[KO13] T. Kobayashi, T. Oshima: *Finite multiplicity theorems for induction and restriction*, Advances in Mathematics, **248**, 921-944, (2013).

[KoSp14] T. Kobayashi, B. Speh: *Intertwining operators and the restriction of representations of rank-one orthogonal groups,* C. R. Math. Acad. Sci. Paris **352**, no. 2, 89-94 (2014).

[KoSp15] T. Kobayashi, B. Speh: *Symmetry breaking for representations of rank one orthogonal groups*, Memoirs of American Mathematical Society **238**, n. 1126, 131 pp (2015).

[KS93] B.Kostant, S. Sahi:*Jordan algebras and Capelli identities.* Invent. Math. **112** , no. 3, 657–664 (1993).

[KrSch] B. Kroetz, H. Schlichtkrull: *Multiplicity bounds and the subrepresentation theorem for real spherical spaces*, arXiv:1309.0930 [math.RT], to appear in Trans. Amer. Math. Soc.

[KV77] M. Kashiwara, M. Vergne: *Remarque sur la covariance de certains opérateurs différentiels.* (French) Non-commutative harmonic analysis (Actes Colloq., Marseille-Luminy, 1976), pp. 119–137. Lecture Notes in Math., Vol. 587, Springer, Berlin, (1977).

[OS07] O. Offen, E. Sayag: *On unitary representations of* GL_{2n} *distinguished by the symplectic group*, *J. Number Theory*, 125(2):344–355, (2007).

[OS08a] O. Offen, E. Sayag: *Global mixed periods and local Klyachko models for the general linear group. Int. Math. Res. Not. IMRN*, (1):Art. ID rnm 136, 25, (2008).

[OS08b] O. Offen, E. Sayag: *Uniqueness and disjointness of Klyachko models. J. Funct. Anal.*, 254(11):2846–2865, (2008).

[OS09] O. Offen, E. Sayag: *The SL(2)-type and base change.* Representation Theory **13**, 228-235 (2009).

[OrSp08] B. Orsted, B. Speh:*Branching laws for some unitary representations of SL(4,R)*, SIGMA 4, 017, 19 pp (2008).

[Sah95] S. Sahi: *Jordan algebras and degenerate principal series*, J. Reine Angew. Math. **462**, 1-18 (1995).

[SaSt90] S. Sahi, E. Stein: *Analysis in matrix space and Speh's representations,* Invent. Math. **101**, no. 2, 379–393 (1990).

[SV] Y. Sakellaridis, A. Venkatesh: *Periods and harmonic analysis on spherical varieties,* arXiv:1203.0039v2

[Say] E. Sayag: $(\mathrm{GL}(2n,\mathbb{C}),\mathrm{Sp}(2n,\mathbb{C}))$ *is a Gelfand pair*, arXiv:0805.2625.

[Sp83] B. Speh: *Unitary representations of* $GL(n,\mathbb{R})$ *with nontrivial* $(\mathfrak{g},K)$*-cohomology* , Invent. Math. 71, 443-465 (1983).

[SZ12] B. Sun, C.-B. Zhu: *Multiplicity one theorems: the archimedean case*, Annals of Mathematics **175**, n. 1, 23-44 (2012).

[Tad86] M. Tadic: *Classification of unitary representations in irreducible representations of general linear group (non-Archimedean case).* Ann. Sci. Ecole Norm. Sup. (4) **19** , no. 3, 335–382 (1986).

[vD86] G. van Dijk: *On a class of generalized Gelfand pairs*, Math. Z. **193**, 581-593 (1986).

[Ven10] A. Venkatesh: *Sparse equidistribution problems, period bounds and subconvexity,* Annals of Mathematics**172**, 989-1094 (2010).

[Vog86] D. A. Vogan: *The unitary dual of GL(n) over an Archimedean field,* Invent. Math. **83**, 449-505 (1986).

[Wall88] N. Wallach: *Real Reductive groups I*, Pure and Applied Math. **132**, Academic Press, Boston, MA (1988).

[Wall92] N. Wallach: *Real Reductive groups II* , Pure and Applied Math. **132-II**, Academic Press, Boston, MA (1992).

[Wald12] J.-L. Waldspurger: *La conjecture locale de Gross-Prasad pour les representations temperees des groupes speciaux orthogonaux.* Asterisque **347**, pp 103-165 (2012).

DMITRY GOUREVITCH, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, WEIZMANN INSTITUTE OF SCIENCE, POB 26, REHOVOT 76100, ISRAEL

E-mail address: dimagur@weizmann.ac.il

URL: http://www.wisdom.weizmann.ac.il/~dimagur

SIDDHARTHA SAHI, DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, HILL CENTER - BUSCH CAMPUS, 110 FREL-INGHUYSEN ROAD PISCATAWAY, NJ 08854-8019, USA

E-mail address: sahi@math.rugers.edu

EITAN SAYAG, DEPARTMENT OF MATHEMATICS, BEN GURION UNIVERSITY OF THE NEGEV, P.O.B. 653, BE'ER SHEVA 84105, ISRAEL

E-mail address: eitan.sayag@gmail.com