Generalized functions - Exercise 1

To be handed in by April 14 in class, or by Moodle.

Solve the following exercises. Questions marked with (*) are optional. For some additional background you may see Terence Tao's notes, available at
`https://terrytao.wordpress.com/2009/04/19/245c-notes-3-distributions`.

1. Let $f \in L^1_{\text{loc}}(\mathbb{R})$, show that ξ_f defined by $\langle \xi_f, g \rangle = \int\limits_{-\infty}^{\infty} fg dx$ for $g \in C_c^\infty(\mathbb{R})$ is a distribution.

2. Find a sequence of continuous functions $\{f_n\}_{n=1}^\infty$ such that f_n converges weakly to f, but does not converge pointwise to f.

3. Find a continuous function f on $\mathbb{R}$ such that $f'' = \delta_0$ as distributions.

4. Let ξ_1 and ξ_2 be distributions. Show that,
   a) $\text{supp}(a\xi_1 + b\xi_2) \subseteq \text{supp}(\xi_1) \cup \text{supp}(\xi_1)$.
   b)* $\text{supp}(\xi_1) - \text{supp}(\xi_1)^\circ \subseteq \text{supp}(\xi_1') \subseteq \text{supp}(\xi_1)$.

5. Let ξ be a compactly supported distribution and $f \in C^\infty(\mathbb{R})$, show that $\xi * f$ is smooth.

6. Let A be a differential operator with constant coefficients and let G_A be the Green function with respect to A, i.e. $A(G_A) = \delta_0$. Show that $G_A * g$ solves the equation $A(f) = g$ for every $g \in C_c^\infty(\mathbb{R})$.

7. * Show that convolution of distributions is associative, that is for $\xi_1, \xi_2, \xi_3 \in C_c^\infty(\mathbb{R})^*$ we have that $\xi_1 * (\xi_2 * \xi_3) = (\xi_1 * \xi_2) * \xi_3$.

8. (Exercise 4 (i) of Terence Tao's notes.) Recall that for $f \in C_c^k(\mathbb{R})$, we define the C^k norm by $\|f\|_{C^k} = \sup\limits_{x \in \mathbb{R}} \sum\limits_{j=0}^{k} |f^{(i)}(x)|$. Show that for a compact $K \subseteq \mathbb{R}$ the functional $\xi : C_c^\infty(K) \to \mathbb{R}$ is continuous if and only if there exist $k \geq 0$ and $C > 0$ such that for all $f \in C_c^\infty(K)$

$$|\langle \xi, f \rangle| \leq C\|f\|_{C^k}.$$

9. Show that all the generalized functions $\xi \in C^{-\infty}(\mathbb{R})$ which are supported on $\{0\}$ are of the form $\sum\limits_{i=0}^{n} c_i \delta^{(i)}$ for some $n \in \mathbb{N}$ and $c_i \in \mathbb{R}$.
   Hint: prove this in three steps.

   (i) Show that there exists n such that ξ is bounded on the set

$$B = \{f \mid f^{(i)}(x) < 1 \forall x \in \mathbb{R}, \forall i < n\}$$

   (ii) Show that there exists $k \in \mathbb{N}$ such that $\xi x^k = 0$, that is $\langle \xi x^k, f \rangle = \langle \xi, x^k f \rangle = 0$ for every $f \in C_c^\infty(\mathbb{R})$.

   (iii) From $\xi x^k = 0$ deduce that $\xi = \sum\limits_{i=0}^{k-1} c_i \delta_0^{(i)}$ for some $c_i \in \mathbb{R}$.