Generalized functions - Exercise 2

Solve the following exercises. Questions marked with (*) are optional.

1. Show that for every neighborhood U of 0 there exists an open balanced set W such that $0 \in W \subseteq U$.

2. Find a topological vector space which is not locally convex (not necessarily of finite dimension).

3. Prove that V is Hausdorff $\iff$ $\{0\}$ is a closed set.

4. Show that if V is finite dimensional and Hausdorff then it is isomorphic to $\mathbb{R}^n$.

1. Show that for a locally convex, complete topological vector space V the following three conditions are equivalent, thus each implying that V is a Fréchet space.

   (a) V is metrizable.

   (b) V is first countable.

   (c) There is a countable collection of semi-norms $\{n_i\}_{i\in\mathbb{N}}$ that defines the basis for the topology over V.

   (Hint: show (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). For (3) $\Rightarrow$ (1): given a countable family of semi-norms define a metric by

   $$d(x,y) := \sum_{k=1}^{\infty} \frac{||x-y||_k}{(1+||x-y||_k)k^2}$$

2. Let V be a topological vector space, show that for every neighborhood U of 0 there exists an open balanced set W such that $0 \in W \subseteq U$.

3. Let $0 \in C$ be an open convex set in a topological vector space V. Show that if C is balanced then $N_C(x)$ is a semi-norm.

4. Find a locally convex topological vector space V such that V has no continuous norm on it. (Hint: we know semi-norms correspond to convex balanced sets. Show that such a semi-norm is a norm if and only if the convex set contains a line?)

5. Show that $C^\infty(\mathbb{R})$ is a complete space. (Hint: show it is locally convex and first countable and use the fact that for these kind of topological vector spaces completeness is equivalent to sequential completeness, thus it is enough to show each Cauchy sequence converges.)

6. * Let V be a locally convex metrizable space. Prove that V is complete (and consequentially is a Fréchet space) $\iff$ it is sequentially complete.